

COPIES OF THE RANDOM GRAPH

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Abstract

Let $\langle R, \sim \rangle$ be the Rado graph, $\text{Emb}(R)$ the monoid of its self-embeddings, $\mathbb{P}(R) = \{f(R) : f \in \text{Emb}(R)\}$ the set of copies of R contained in R , and \mathcal{I}_R the ideal of subsets of R which do not contain a copy of R . We consider the poset $\langle \mathbb{P}(R), \subset \rangle$, the algebra $\mathbb{P}(R)/\mathcal{I}_R$, and the inverse of the right Green's pre-order on $\text{Emb}(R)$, and show that these pre-orders are forcing equivalent to a two step iteration of the form $\mathbb{P} * \pi$, where the poset \mathbb{P} is similar to the Sacks perfect set forcing: adds a generic real, has the \aleph_0 -covering property and, hence, preserves ω_1 , has the Sacks property and does not produce splitting reals, while π codes an ω -distributive forcing. Consequently, the Boolean completions of these four posets are isomorphic and the same holds for each countable graph containing a copy of the Rado graph.

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1 Introduction

In this paper we continue the investigation of the partial orderings of the form $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$, where \mathbb{X} is an ultrahomogeneous relational structure and $\mathbb{P}(\mathbb{X})$ the set of domains of substructures of \mathbb{X} isomorphic to \mathbb{X} . In particular, if $\mathbb{X} = \langle X, \rho \rangle$ is a binary structure (that is $\rho \subset X \times X$), then $\mathbb{P}(\mathbb{X}) = \{A \subset X : \langle A, \rho_A \rangle \cong \langle X, \rho \rangle\}$, where $\rho_A = \rho \cap (A \times A)$. In the sequel, in order to simplify notation, instead of $\langle \mathbb{P}(\mathbb{X}), \subset \rangle$ we will write $\mathbb{P}(\mathbb{X})$ whenever the context admits.

This investigation is related to a coarse classification of relational structures. Namely, the conditions $\mathbb{P}(\mathbb{X}) = \mathbb{P}(\mathbb{Y})$, $\mathbb{P}(\mathbb{X}) \cong \mathbb{P}(\mathbb{Y})$, $\text{sq } \mathbb{P}(\mathbb{X}) \cong \text{sq } \mathbb{P}(\mathbb{Y})$ and $\text{ro sq } \mathbb{P}(\mathbb{X}) \cong \text{ro sq } \mathbb{P}(\mathbb{Y})$ (where $\text{sq } \mathbb{P}$ denotes the separative quotient of a partial order \mathbb{P} and $\text{ro sq } \mathbb{P}$ its Boolean completion) define different equivalence relations (“similarities”) on the class of relational structures and their interplay with the similarities defined by the conditions $\mathbb{X} = \mathbb{Y}$, $\mathbb{X} \cong \mathbb{Y}$ and $\mathbb{X} \rightleftarrows \mathbb{Y}$ (equimorphism) was considered in [11]. It turns out that the similarity defined by the condition $\text{ro sq } \mathbb{P}(\mathbb{X}) \cong \text{ro sq } \mathbb{P}(\mathbb{Y})$ is implied by all the similarities listed above and, thus,

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provides the coarsest among the mentioned classifications of relational structures. Since the posets of copies are always homogeneous, the condition $\text{ro sq } \mathbb{P}(\mathbb{X}) \cong \text{ro sq } \mathbb{P}(\mathbb{Y})$ is equivalent to the forcing equivalence of the posets $\mathbb{P}(\mathbb{X})$ and $\mathbb{P}(\mathbb{Y})$ (we will write $\mathbb{P}(\mathbb{X}) \equiv \mathbb{P}(\mathbb{Y})$) and, for convenience, we will exploit this fact using the tools of set-theoretic forcing in our proofs.

This paper can also be regarded as a part of the investigation of the quotient algebras of the form $P(\omega)/\mathcal{I}$, where \mathcal{I} is an ideal on ω . Namely, by [8], if \mathbb{X} is a countable indivisible structure with domain ω , then the collection $\mathcal{I}_{\mathbb{X}}$ of subsets of ω which do not contain a copy of \mathbb{X} is either the ideal of finite sets or a co-analytic tall ideal and the poset $\text{sq } \mathbb{P}(\mathbb{X})$ is isomorphic to a dense subset of $(P(\omega)/\mathcal{I}_{\mathbb{X}})^+$, which implies $\text{ro sq } \mathbb{P}(\mathbb{X}) \cong \text{ro}(P(\omega)/\mathcal{I}_{\mathbb{X}})^+$. So, since the structure considered in this paper, the Rado graph, $\langle R, \sim \rangle$, is indivisible, our results can be regarded as statements concerning the forcing related properties of the corresponding quotient algebra. Namely, if we call a graph scattered if it does not contain a copy of the Rado graph, and if \mathcal{I}_R denotes the ideal of scattered subgraphs of R , then

$$\text{ro sq } \mathbb{P}(R) = \text{ro}((P(R)/\mathcal{I}_R)^+).$$

As a consequence of the main result of [15] we have the following statement describing the forcing related properties of the poset of copies of the rational line, \mathbb{Q} , and the corresponding quotient $P(\mathbb{Q})/\text{Scatt}$, where Scatt denotes the ideal of scattered suborders of \mathbb{Q} . Namely, if \mathbb{S} denotes the Sacks perfect set forcing and $sh(\mathbb{S})$ the size of the continuum in the Sacks extension, then we have

Theorem 1.1 *For each countable non-scattered linear order L and, in particular, for the rational line, the poset $\mathbb{P}(L)$ is forcing equivalent to the two-step iteration*

$$\mathbb{S} * \pi,$$

where $1_{\mathbb{S}} \Vdash \text{“}\pi \text{ is a } \sigma\text{-closed forcing”}$. If the equality $sh(\mathbb{S}) = \aleph_1$ (implied by CH) or PFA holds in the ground model, then the second iterand is forcing equivalent to the poset $(P(\omega)/\text{Fin})^+$ of the Sacks extension. Consequently,

$$\text{ro sq } \mathbb{P}(\mathbb{Q}) \cong \text{ro}((P(\mathbb{Q})/\text{Scatt})^+) \cong \text{ro}(\mathbb{S} * \pi).$$

(We note that by [9] the poset of copies of a countable *scattered* linear order L is forcing equivalent to a separative atomless ω_1 -closed poset; thus, under CH, to $(P(\omega)/\text{Fin})^+$ and then $\text{ro sq } \mathbb{P}(L) \cong \text{ro}(P(\omega)/\text{Fin})^+$. The posets of copies of countable ordinals are described in [10].)

In this paper we prove a similar statement for non-scattered graphs (that is, the graphs containing a copy of the Rado graph):

Theorem 1.2 *For each countable non-scattered graph $\langle G, \sim \rangle$ and, in particular, for the Rado graph, the poset $\mathbb{P}(G)$ is forcing equivalent to the two-step iteration*

$$\mathbb{P} * \pi,$$

where $1_{\mathbb{P}} \Vdash \text{“}\pi \text{ is an } \omega\text{-distributive forcing”}$ and the poset \mathbb{P} is similar to the Sacks forcing: adds a generic real, has the \aleph_0 -covering property (thus preserves ω_1), has the Sacks property and does not produce splitting reals. In addition,

$$\text{ro sq } \mathbb{P}(G) \cong \text{ro}(P(R)/\mathcal{I}_R)^+ \cong \text{ro}(\mathbb{P} * \pi)$$

and these complete Boolean algebras are weakly distributive³.

In fact, if $\langle G, \sim \rangle$ is a countable graph containing a copy of the Rado graph, then these two structures are equimorphic and, by [11], forcing equivalent. So it is sufficient to prove the previous theorem assuming that $\langle G, \sim \rangle$ is the Rado graph.

Finally we note that the results of this paper are related to the investigation of the monoids of self-embeddings. We recall that the right Green’s pre-order \preceq^R on a monoid $\langle M, \cdot, 1 \rangle$ is defined by $x \preceq^R y$ iff $x \cdot z = y$, for some z . It is easy to check (see [12]) that the poset of copies $\mathbb{P}(\mathbb{X})$ of a structure \mathbb{X} is isomorphic to the antisymmetric quotient of the pre-order $\langle \text{Emb}(\mathbb{X}), (\preceq^R)^{-1} \rangle$ and, consequently, these pre-orders are forcing equivalent. Thus, by Theorem 1.2, for the Rado graph we have $\langle \text{Emb}(R), (\preceq^R)^{-1} \rangle \equiv (\mathbb{P} * \pi)$ and the Boolean completion of the pre-order $\langle \text{Emb}(R), (\preceq^R)^{-1} \rangle$ is a weakly distributive complete Boolean algebra.

2 Preliminaries

First we introduce a convenient notation. If $\langle G, \sim \rangle$ is a graph (namely, if \sim is a symmetric and irreflexive binary relation on the set G) and $K \subset H \in [G]^{<\omega}$, let

$$G_K^H := \left\{ v \in G \setminus H : \forall k \in K (v \sim k) \wedge \forall h \in H \setminus K (v \not\sim h) \right\}.$$

(Clearly, $G_\emptyset^\emptyset = G$.)

The object of our study is the Rado graph (the Erdős-Rényi graph, the countable random graph) introduced independently by Erdős and Rényi [2] and Rado [17].

³A complete Boolean algebra \mathbb{B} is called *weakly distributive* (or $(\omega, \cdot, <\omega)$ -distributive) iff for each cardinal κ and each matrix $[b_{n\alpha} : \langle n, \alpha \rangle \in \omega \times \kappa]$ of elements of \mathbb{B} we have

$$\bigwedge_{n \in \omega} \bigvee_{\alpha \in \kappa} b_{n\alpha} = \bigvee_{s: \omega \rightarrow [\kappa]^{<\omega}} \bigwedge_{n \in \omega} \bigvee_{\alpha \in s(n)} b_{n\alpha}.$$

It is characterized as the unique (up to isomorphism) countable graph $\langle R, \sim \rangle$ such that

$$R_K^H \neq \emptyset, \text{ whenever } K \subset H \in [R]^{<\omega}. \quad (1)$$

Equivalently, the Rado graph can be characterized as the unique countable ultrahomogeneous universal graph (see [5]) or as the Fraïssé limit of the amalgamation class of all finite graphs (see [3]). In addition, by [2], if a graph with countably many vertices is chosen at random, by picking edges independently with probability $\frac{1}{2}$, then, with probability 1, the obtained graph will be isomorphic to the Rado graph. The Rado graph and several related structures (for example the automorphism group and the endomorphism monoid of $\langle R, \sim \rangle$, various topologies on R etc.) were extensively explored (see the survey article [1]). The following fact contains the basic properties of the Rado graph which will be used in the paper.

Fact 2.1 *Let $\langle R, \sim \rangle$ be a Rado graph and $\mathbb{P}(R)$ the set of its copies. Then*

- (a) *If F is a finite subset of R , then $R \setminus F \in \mathbb{P}(R)$;*
- (b) *If $\{X_1, \dots, X_k\}$ is a partition of R , then $X_i \in \mathbb{P}(R)$, for some $i \leq k$ (the Rado graph is a strongly indivisible structure);*
- (c) *If H is a finite subset of R , then $\{H\} \cup \{R_K^H : K \subset H\}$ is a partition of R and $R_K^H \in \mathbb{P}(R)$, for each $K \subset H$.*

Concerning the order theoretic properties of the poset $\langle \mathbb{P}(R), \subset \rangle$ we note that it is a homogeneous, atomless and chain complete suborder of the order $\langle [R]^\omega, \subset \rangle$ having a largest element, R . In addition, by [14], it contains maximal antichains of size \mathfrak{c} , \aleph_0 and n , for each positive integer n , and in [13] the order types of maximal chains in this poset are characterized as the order types of sets of the form $K \setminus \{\min K\}$, where K is a compact subset of the real line having the minimum non-isolated.

The sets R_K^H (the orbits of R) will play an important role in our constructions.

Lemma 2.2 *Let H_1 and H_2 be finite subsets of R , $K_1 \subset H_1$ and $K_2 \subset H_2$. Then*

- (a) $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} \neq \emptyset$ if and only if $H_1 \cap K_2 = H_2 \cap K_1$;
- (b) $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} \neq \emptyset$ implies that $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} = R_{K_1 \cup K_2}^{H_1 \cup H_2}$;
- (c) $R_{K_1}^{H_1} = R_{K_2}^{H_2}$ if and only if $H_1 = H_2$ and $K_1 = K_2$;
- (d) $R_{K_1}^{H_1} \subset R_{K_2}^{H_2}$ if and only if $H_1 \supset H_2$, $K_1 \supset K_2$ and $H_2 \cap K_1 = K_2$.

Proof. We prove (a) and (b) simultaneously.

(\Rightarrow) Assuming that $v \in R_{K_1}^{H_1} \cap R_{K_2}^{H_2}$ we first show that $H_1 \cap K_2 \subset H_2 \cap K_1$. If $r \in H_1 \cap K_2$, then $r \in H_2$ and, since $v \in R_{K_2}^{H_2}$, we have $v \sim r$. Now $r \notin K_1$ would imply $r \in H_1 \setminus K_1$ and, since $v \in R_{K_1}^{H_1}$, we would have $v \not\sim r$, which is not true. So $r \in K_1$ and we are done. The reversed inclusion has a symmetric proof.

(\Leftarrow) Let $H_1 \cap K_2 = H_2 \cap K_1$. Since $H_1 = (H_1 \setminus H_2) \dot{\cup} (H_1 \cap H_2)$ and $K_1 = (K_1 \setminus H_2) \dot{\cup} (K_1 \cap H_2)$ and the second partition refines the first we have

$$R_{K_1}^{H_1} = R_{K_1 \setminus H_2}^{H_1 \setminus H_2} \cap R_{K_1 \cap H_2}^{H_1 \cap H_2}. \quad (2)$$

Similarly, $H_2 = (H_2 \setminus H_1) \dot{\cup} (H_2 \cap H_1)$ and $K_2 = (K_2 \setminus H_1) \dot{\cup} (K_2 \cap H_1)$, thus

$$R_{K_2}^{H_2} = R_{K_2 \setminus H_1}^{H_2 \setminus H_1} \cap R_{K_2 \cap H_1}^{H_2 \cap H_1}. \quad (3)$$

Now since $\{H_1 \setminus H_2, H_1 \cap H_2, H_2 \setminus H_1\}$ is a partition of the set $H_1 \cup H_2$ and, by the assumption, $\{K_1 \setminus H_2, K_1 \cap H_2, K_2 \setminus H_1\}$ a partition of the set $K_1 \cup K_2$ refining the mentioned partition of $H_1 \cup H_2$, we have

$$R_{K_1}^{H_1} \cap R_{K_2}^{H_2} = R_{K_1 \setminus H_2}^{H_1 \setminus H_2} \cap R_{K_1 \cap H_2}^{H_1 \cap H_2} \cap R_{K_2 \setminus H_1}^{H_2 \setminus H_1} = R_{K_1 \cup K_2}^{H_1 \cup H_2}. \quad (4)$$

and by Fact 2.1(c), $R_{K_1 \cup K_2}^{H_1 \cup H_2} \neq \emptyset$ (moreover, this set is a copy of R).

(c) Let $R_{K_1}^{H_1} = R_{K_2}^{H_2}$. Suppose that $H_2 \setminus H_1 \neq \emptyset$ and let $v \in H_2 \setminus H_1$. If $v \in K_2$, then, since $v \notin H_1$, there is $w \in R_{K_1}^{H_1} \cap R_{\emptyset}^{\{v\}}$, thus $w \not\sim v$. But $w \in R_{K_2}^{H_2}$ and, since $v \in K_2$ we have $w \sim v$, so we have a contradiction. Otherwise, if $v \in H_2 \setminus K_2$, there is $w \in R_{K_1}^{H_1} \cap R_{\{v\}}^{\{v\}}$, thus $w \sim v$. But $w \in R_{K_2}^{H_2}$ and, since $v \notin K_2$ we have $w \not\sim v$ and we get a contradiction again. Thus $H_2 \subset H_1$ and, similarly, $H_1 \subset H_2$. So $H_1 = H_2$, which by (a) implies $H_1 \cap K_2 = H_1 \cap K_1$, that is $K_2 = K_1$.

(d) If $R_{K_1}^{H_1} \subset R_{K_2}^{H_2}$, then by (b) we have $R_{K_1}^{H_1} = R_{K_1}^{H_1} \cap R_{K_2}^{H_2} = R_{K_1 \cup K_2}^{H_1 \cup H_2}$ which, by (c), implies $H_1 = H_1 \cup H_2$ and $K_1 = K_1 \cup K_2$, that is $H_1 \supset H_2$ and $K_1 \supset K_2$. Hence $H_1 \cap K_2 = K_2$ so, by (a), $H_2 \cap K_1 = K_2$.

If $H_1 \supset H_2$, $K_1 \supset K_2$ and $H_2 \cap K_1 = K_2$, then, since $K_2 = H_1 \cap K_2$, by (a) and (b) we have $R_{K_1}^{H_1} \cap R_{K_2}^{H_2} = R_{K_1 \cup K_2}^{H_1 \cup H_2} = R_{K_1}^{H_1}$, that is $R_{K_1}^{H_1} \subset R_{K_2}^{H_2}$. \square

The reader will notice that, by (c) and (d) of the previous lemma, the mapping $F : \text{Fn}(\omega, 2) \rightarrow \mathbb{P}(R)$ given by $F(\varphi) = R_{\varphi^{-1}[\{1\}]}^{\text{dom } \varphi}$ is an embedding of the Cohen poset $\langle \text{Fn}(\omega, 2), \supset \rangle$ into the poset $\langle \mathbb{P}(R), \subset \rangle$. But F is not a dense embedding (we recall that $\mathbb{P}(R)$ contains antichains of size \mathfrak{c}) and this fact does not imply that the poset $\mathbb{P}(R)$ is forcing equivalent to the Cohen forcing.

3 Labeling of the vertices of the Rado graph

Let $\langle R, \sim \rangle$ be the Rado graph. A *labeling* of $L \in \mathbb{P}(R)$ is a pair $\mathcal{L} = \langle \Pi, q \rangle$, where

- (L1) $\Pi = \{L_n : n \in \omega\}$ is a partition of the set L ,
- (L2) $q : \bigcup_{n \in \omega} \{n\} \times P(\bigcup_{i < n} L_i) \rightarrow L$ is a bijection,
- (L3) $L_n = \{q(n, K) : K \subset \bigcup_{i < n} L_i\}$, for each $n \in \omega$,

(L4) $q(n, K) \in L_K^{\bigcup_{i < n} L_i}$, for each $n \in \omega$ and each $K \subset \bigcup_{i < n} L_i$.

Then, clearly, $L_0 = \{q(0, \emptyset)\}$, $|L_0| = 1$ and the sets L_n are finite. More precisely, by (L3) we have $|L_n| = m_n$, where the integers m_n , $n \in \omega$, are defined by: $m_0 = 1$ and $m_n = 2^{\sum_{i < n} m_i}$, for $n > 0$. Thus $\langle |L_n| : n \in \omega \rangle = \langle 1, 2, 8, 2^{11}, \dots \rangle$.

Lemma 3.1 *Each copy L of R has infinitely many labelings.*

Proof. Let \prec_0 be a well ordering on L such that $\langle L, \prec_0 \rangle \cong \langle \omega, < \rangle$, where $<$ is the natural ordering on ω ; in fact w.l.o.g. we can assume that $\langle L, \prec_0 \rangle = \langle \omega, < \rangle$. By recursion we define a sequence $\langle L_n : n \in \omega \rangle$ such that for each $m, n \in \omega$ we have

- (i) L_n is a finite subset of L ,
- (ii) $L_m \cap L_n = \emptyset$, if $m \neq n$,
- (iii) $L_n = \{\min L_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$.

First, since $L_\emptyset = L$, the sequence $\langle L_0 \rangle$, where $L_0 = \{0\}$ satisfies (i), (ii) and (iii).

If $n > 0$ and if $\langle L_i : i < n \rangle$ is a sequence satisfying (i) - (iii), then $\bigcup_{i < n} L_i$ is a finite subset of L and, by Fact 2.1(c), for $K \subset \bigcup_{i < n} L_i$ we have $L_K^{\bigcup_{i < n} L_i} \neq \emptyset$. Thus we define L_n by (iii) and, since $(\bigcup_{i < n} L_i) \cap L_K^{\bigcup_{i < n} L_i} = \emptyset$, the extended sequence $\langle L_i : i < n + 1 \rangle$ satisfies (i) - (iii). The recursion works.

Suppose that there is $m \in L \setminus \bigcup_{n \in \omega} L_n$. For $n \in \omega$, since $m \notin \bigcup_{i < n} L_i$, by Fact 2.1(c) there is $K_n \subset \bigcup_{i < n} L_i$ such that $m \in L_{K_n}^{\bigcup_{i < n} L_i}$ and, since $m \notin L_n$, by (iii) we have $m > \min L_{K_n}^{\bigcup_{i < n} L_i} \in L_n$. Thus for each $n \in \omega$ there is $q \in L_n$ such that $m > q$ and, by (ii), m is greater than infinitely many natural numbers, which is impossible. So, $\Pi := \{L_n : n \in \omega\}$ is a partition of the set L .

Let the mapping $q : \bigcup_{n \in \omega} \{n\} \times P(\bigcup_{i < n} L_i) \rightarrow L$ be defined by $q(n, K) = \min L_K^{\bigcup_{i < n} L_i}$. Since $L = \bigcup_{n \in \omega} L_n$ the mapping q is a surjection. If $q(n, K) = q(n', K')$, then, since $q(n, K) \in L_n$, by (ii) we have $n = n'$, which by Fact 2.1(c) implies $K = K'$. Thus q is a bijection, (L3) and (L4) follow from (iii) and $\mathcal{L} = \langle \Pi, q \rangle$ is a labeling of L determined by the well ordering $<$.

Clearly, if $n \in \omega$ and \prec_n is a well ordering on $L = \omega$ such that $\langle L, \prec_n \rangle \cong \langle \omega, < \rangle$ and $\min_{\prec_n} L = n$ then repeating the previous construction using \prec_n instead of \prec_0 we obtain a labeling \mathcal{L}_n of L , for which we have $L_0 = \{n\}$. So the labelings \mathcal{L}_n , $n \in \omega$, are different. \square

For convenience, instead of $q(n, K)$ we will write $q_K^{\bigcup_{i < n} L_i}$ and a labeling will be denoted by

$$\left\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \right\rangle.$$

4 Copies with orbits refining maximal antichains

The following construction of copies of R will be frequently used in the paper. We note that if $A \in \mathbb{P}(R)$ and $K \subset H \in [A]^{<\omega}$, then A with the induced graph structure is a Rado graph, and, clearly, $\mathbb{P}(A) = P(A) \cap \mathbb{P}(R)$ and $A_K^H = A \cap R_K^H$.

Lemma 4.1 *If $A \in \mathbb{P}(R)$ and if $L = \bigcup_{n \in \omega} L_n$, where for each $n \in \omega$ we have $L_n = \{q_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$ and $q_K^{\bigcup_{i < n} L_i} \in A_K^{\bigcup_{i < n} L_i}$, for all $K \subset \bigcup_{i < n} L_i$, then*

- (a) $L \in \mathbb{P}(A)$;
- (b) $\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \rangle$ is a labeling of L ;
- (c) If $\varphi(u, v, w)$ is a formula of the language of set theory, τ is a $\mathbb{P}(R)$ -name and $S_n \in \mathbb{P}(A)$, for $n \in \omega$, where $L \subset S_{n+1} \subset S_n$, for each $n \in \omega$, and if

$$\forall n \in \omega \quad \forall K \subset \bigcup_{i < n} L_i \quad \exists a_K^n \quad (S_n)_{K^{\bigcup_{i < n} L_i}} \Vdash \varphi(\tau, \check{n}, a_K^n), \quad (5)$$

then for each $n \in \omega$ and each $K \subset \bigcup_{i < n} L_i$ we have $L_K^{\bigcup_{i < n} L_i} \Vdash \varphi(\tau, \check{n}, a_K^n)$.

Proof. (a) If $K \subset H$ are finite subsets of L , then $H \subset \bigcup_{i < n} L_i$ for some $n \in \omega$ and $q_K^{\bigcup_{i < n} L_i} \in L \cap A_K^{\bigcup_{i < n} L_i} \subset L \cap R_K^H = L_K^H$. Thus the graph L satisfies (1).

(b) By the assumption, the mapping $q : \bigcup_{n \in \omega} \{n\} \times P(\bigcup_{i < n} L_i) \rightarrow L$ defined by $q(n, K) = q_K^{\bigcup_{i < n} L_i}$ is a surjection. If $q_{K_1}^{\bigcup_{i < n_1} L_i} = q_{K_2}^{\bigcup_{i < n_2} L_i} =: u$, then, since $u \in L_{n_1} \cap A_{K_2}^{\bigcup_{i < n_2} L_i} \subset L_{n_1} \cap R \setminus \bigcup_{i < n_2} L_i$ we have $n_1 \geq n_2$ and, similarly, $n_2 \geq n_1$, which gives $n_1 = n_2$. By Fact 2.1(c), $K_1 \neq K_2$ implies $A_{K_1}^{\bigcup_{i < n_1} L_i} \cap A_{K_2}^{\bigcup_{i < n_1} L_i} = \emptyset$, thus $K_1 = K_2$, and, hence, q is an injection. This implies that $\{L_n : n \in \omega\}$ is a partition of L and conditions (L3) and (L4) are obviously satisfied.

(c) Since $L \subset S_n$ we have $L_K^{\bigcup_{i < n} L_i} \subset (S_n)_K^{\bigcup_{i < n} L_i}$ and we apply (5). \square

Roughly speaking, in order to provide condition (5) it is sufficient that for each maximal antichain \mathcal{A}_n in $\mathbb{P}(R)$ such that each $A \in \mathcal{A}_n$ forces $\varphi(\tau, \check{n}, \check{a})$, for some a , there is $S_n \in \mathbb{P}(S_{n-1})$ containing $\bigcup_{i < n} L_i$ and such that each orbit $(S_n)_K^{\bigcup_{i < n} L_i}$ is contained in some $A \in \mathcal{A}_n$. This follows from the following theorem, the main statement of this section.

Theorem 4.2 *For each maximal antichain \mathcal{A} in the poset $\mathbb{P}(R)$ and each finite set $F_0 \subset R$ there is $S \in \mathbb{P}(R)$ such that $F_0 \subset S$ and*

$$\forall H \subset F_0 \quad \exists A \in \mathcal{A} \quad S_H^{F_0} \subset A. \quad (6)$$

The proof of Theorem 4.2, given at the end of the section, is based on the following three lemmas.

Lemma 4.3 *For each $p \in R$ and each finite (possibly empty) sets $F \subset R_{\{p\}}^{\{p\}}$ and $G \subset R_{\emptyset}^{\{p\}}$ there is an isomorphism $f : R_{\{p\}}^{\{p\}} \rightarrow R_{\emptyset}^{\{p\}}$ satisfying*

$$\forall u, v \in R_{\{p\}}^{\{p\}} \quad (u \sim f(v) \Leftrightarrow f(u) \sim v), \quad (7)$$

$$F \cap f^{-1}[G] = \emptyset. \quad (8)$$

Proof. Let \mathbb{P} be the set of all partial functions φ from $R_{\{p\}}^{\{p\}}$ to $R_{\emptyset}^{\{p\}}$ such that for each $u, v \in \text{dom } \varphi$ we have

- (i) $u \neq v \Rightarrow \varphi(u) \neq \varphi(v)$,
- (ii) $u \sim v \Leftrightarrow \varphi(u) \sim \varphi(v)$,
- (iii) $u \sim \varphi(v) \Leftrightarrow \varphi(u) \sim v$,
- (iv) $u \in F \Rightarrow \varphi(u) \notin G$.

Claim 1. $D_u = \{\varphi \in \mathbb{P} : u \in \text{dom } \varphi\}$, $u \in R_{\{p\}}^{\{p\}}$, are dense sets in $\langle \mathbb{P}, \supset \rangle$.

Proof of Claim 1. Let $u_1 \in R_{\{p\}}^{\{p\}}$ and $\psi \in \mathbb{P} \setminus D_{u_1}$, that is, $u_1 \notin \text{dom } \psi$. Let

$$H = \{u \in \text{dom } \psi : u \sim u_1\} \text{ and } L = \{v \in \text{dom } \psi : \psi(v) \sim u_1\}.$$

Since the sets $\{p\}$, $\text{dom } \psi$ and $\text{ran } \psi$ are pairwise disjoint we choose

$$v_1 \in (R_{\emptyset}^{\{p\}} \cap R_L^{\text{dom } \psi} \cap R_{\psi[H]}^{\text{ran } \psi}) \setminus G \quad (9)$$

and show that $\varphi = \psi \cup \{\langle u_1, v_1 \rangle\} \in \mathbb{P}$. Since $u_1 \notin \text{dom } \psi$, φ is a function and, by (9), we have $v_1 \notin \text{ran } \psi$ which implies (i).

Since $\psi \in \mathbb{P}$ for a proof that φ satisfies (ii) we show that

$$\forall u \in \text{dom } \psi \quad (u \sim u_1 \Leftrightarrow \psi(u) \sim v_1). \quad (10)$$

Let $u \in \text{dom } \psi$. If $u \sim u_1$, then $u \in H$ thus $\psi(u) \in \psi[H]$ and, by (9), $\psi(u) \sim v_1$. If $u \not\sim u_1$, then $u \in \text{dom } \psi \setminus H$ and, hence, $\psi(u) \in \text{ran } \psi \setminus \psi[H]$, which, together with (9), implies $\psi(u) \not\sim v_1$. So (10) is true.

For a proof that φ satisfies (iii) we show that

$$\forall v \in \text{dom } \psi \quad (u_1 \sim \psi(v) \Leftrightarrow v_1 \sim v). \quad (11)$$

Let $v \in \text{dom } \psi$. If $\psi(v) \sim u_1$, then $v \in L$ and, by (9), $v_1 \sim v$. If $\psi(v) \not\sim u_1$, then $v \in \text{dom } \psi \setminus L$ and, by (9), $v_1 \not\sim v$. Thus (11) is true.

Since $\psi \in \mathbb{P}$ and, by (9), $\varphi(u_1) = v_1 \notin G$, φ satisfies (iv). Thus $\varphi \in \mathbb{P}$ and, clearly $\psi \subset \varphi \in D_{u_1}$. Claim 1 is proved. \square

Claim 2. $\Delta_v = \{\varphi \in \mathbb{P} : v \in \text{ran } \varphi\}$, $v \in R_{\emptyset}^{\{p\}}$, are dense sets in $\langle \mathbb{P}, \supset \rangle$.

Proof of Claim 2. Let $v_1 \in R_{\emptyset}^{\{p\}}$ and $\psi \in \mathbb{P} \setminus \Delta_{v_1}$, that is, $v_1 \notin \text{ran } \psi$. Let

$$H = \{v \in \text{ran } \psi : v \sim v_1\} \text{ and } L = \{u \in \text{dom } \psi : u \sim v_1\}. \quad (12)$$

Since the sets $\{p\}$, $\text{dom } \psi$ and $\text{ran } \psi$ are pairwise disjoint we choose

$$u_1 \in (R_{\{p\}}^{\{p\}} \cap R_{\psi^{-1}[H]}^{\text{dom } \psi} \cap R_{\psi[L]}^{\text{ran } \psi}) \setminus F \quad (13)$$

and show that $\varphi = \psi \cup \{\langle u_1, v_1 \rangle\} \in \mathbb{P}$. By (13) we have $u_1 \notin \text{dom } \psi$ so φ is a function and, since $v_1 \notin \text{ran } \psi$, φ satisfies (i).

Since $\psi \in \mathbb{P}$, for a proof that φ satisfies (ii) it remains to be shown that (10) holds. Let $u \in \text{dom } \psi$. If $u \sim u_1$, then, by (13), $u \in \psi^{-1}[H]$ thus $\psi(u) \in H$ and, hence, $\psi(u) \sim v_1$. If $u \not\sim u_1$, then $u \in \text{dom } \psi \setminus \psi^{-1}[H]$ and, hence, $\psi(u) \in \text{ran } \psi \setminus H$, which, by (12), implies $\psi(u) \not\sim v_1$ and (10) is true.

For a proof of (iii) we verify (11). Let $v \in \text{dom } \psi$. If $v \sim v_1$, then $v \in L$ and, hence, $\psi(v) \in \psi[L]$ so, by (13), $u_1 \sim \psi(v)$. If $v \not\sim v_1$, then $v \in \text{dom } \psi \setminus L$ and, hence, $\psi(v) \in \text{ran } \psi \setminus \psi[L]$ and, by (13), $u_1 \not\sim \psi(v)$. Thus (11) is true.

Since $\psi \in \mathbb{P}$ and, by (13), $u_1 \notin F$, φ satisfies (iv).

Thus $\varphi \in \mathbb{P}$ and, clearly $\psi \subset \varphi \in \Delta_{v_1}$. Claim 2 is proved. \square

By Claims 1, 2 and the Rasiowa-Sikorski theorem there is a filter \mathcal{G} in the poset $\langle \mathbb{P}, \supset \rangle$ intersecting the sets D_u , $u \in R_{\{p\}}^{\{p\}}$, and Δ_v , $v \in R_{\emptyset}^{\{p\}}$. Thus $f = \bigcup_{\varphi \in \mathcal{G}} \varphi \subset R_{\{p\}}^{\{p\}} \times R_{\emptyset}^{\{p\}}$ and $\text{dom } f = R_{\{p\}}^{\{p\}}$ and $\text{ran } f = R_{\emptyset}^{\{p\}}$. So, since \mathcal{G} is a set of compatible functions, f is a surjection from $R_{\{p\}}^{\{p\}}$ onto $R_{\emptyset}^{\{p\}}$. By (i) f is an injection, by (ii) it is a graph-isomorphism, by (iii) satisfies (7) and, by (iv), satisfies (8). \square

Let $p \in R$ and let $F \subset R_{\{p\}}^{\{p\}}$ and $G \subset R_{\emptyset}^{\{p\}}$ be finite (possibly empty) sets. A set $C \subset R_{\{p\}}^{\{p\}}$ will be called (p, F, G) -*extendible* iff there is a set $C' \subset R_{\emptyset}^{\{p\}}$ such that $F \cup G \subset C \cup \{p\} \cup C' \in \mathbb{P}(R)$. (Then, by Fact 2.1(c), C and C' are copies of R .) $(p, \emptyset, \emptyset)$ -extendible copies will be called p -*extendible*.

For $F = G = \emptyset$, the following statement shows that there is a copy $B \subset R_{\{p\}}^{\{p\}}$ such that the set of p -extendible copies is dense below B . Moreover we have

Lemma 4.4 *Let $p \in R$, let $F \subset R_{\{p\}}^{\{p\}}$ and $G \subset R_{\emptyset}^{\{p\}}$ be finite (possibly empty) sets and $f : R_{\{p\}}^{\{p\}} \rightarrow R_{\emptyset}^{\{p\}}$ an isomorphism satisfying (7) and (8). Then there is a*

copy $B \in \mathbb{P}(R_{\{p\}}^{\{p\}})$ such that $F \cup f^{-1}[G] \subset B$ and that for each set A satisfying

$$F \cup f^{-1}[G] \subset A \in \mathbb{P}(B) \quad (14)$$

there are sets A_0 and A_1 such that

$$A_0 \cup A_1 \subset A \quad \wedge \quad A_0 \cap A_1 = \emptyset \quad \wedge \quad F \subset A_0 \quad \wedge \quad G \subset f[A_1], \quad (15)$$

$$A_0 \cup \{p\} \cup f[A_1] \in \mathbb{P}(R). \quad (16)$$

Thus A_0 is a (p, F, G) -extendible copy contained in A .

Proof. Let $B = \bigcup_{n \in \omega} L_n$, where $L_0 = F$, $L_1 = f^{-1}[G]$ and, for $n \geq 1$,

$$L_{n+1} = \{q_K^{\bigcup_{k \leq n} L_k} : K \subset \bigcup_{k \leq n} L_k\}, \text{ where} \quad (17)$$

$$q_K^{\bigcup_{k \leq n} L_k} \in R_{\{p\}}^{\{p\}} \cap R_K^{\bigcup_{k \leq n} L_k} \cap R_{f[K]}^{f[\bigcup_{k \leq n} L_k]}, \text{ for each } K \subset \bigcup_{k \leq n} L_k. \quad (18)$$

Then $F \cup f^{-1}[G] \subset B \subset R_{\{p\}}^{\{p\}}$ and, as in Lemma 4.1, we show that $B \in \mathbb{P}(R)$.

Let A be a set satisfying (14). We will construct sets A_0 and A_1 satisfying (15) and (16).

First by recursion we construct finite sets $S_{i,j} \subset \omega \setminus \{0, 1\}$, for $2 \leq i < \omega$ and $j \in \{0, 1\}$, and $a_n \in A$, for $n \in \bigcup_{2 \leq i < \omega} \bigcup_{j < 2} S_{i,j}$, such that

- (i) $a_n \in A \cap L_n$, for $n \in \bigcup_{2 \leq i < \omega} \bigcup_{j < 2} S_{i,j}$,
- (ii) $\langle i, j \rangle <_{lex} \langle i_1, j_1 \rangle$ implies $S_{i,j} < S_{i_1, j_1}$ (that is, $\max S_{i,j} < \min S_{i_1, j_1}$),
- (iii) For each $i_0 \geq 2$, each $K \subset L_0 \cup L_1 \cup A_{< i_0}$ (where, for simplicity, we define $S_{< i_0} := \bigcup_{2 \leq i < i_0} \bigcup_{j < 2} S_{i,j}$ and $A_{< i_0} := \{a_n : n \in S_{< i_0}\}$, thus $S_{< 2} = A_{< 2} = \emptyset$) and each $j_0 < 2$ there is n such that

$$n \in S_{i_0, j_0}, \quad (19)$$

$$a_n \in R_K^{L_0 \cup L_1 \cup A_{< i_0}}. \quad (20)$$

Claim 0. The recursion works.

Proof of Claim 0. Let $i_0 \geq 2$ and let $\langle S_{i,j} : 2 \leq i < i_0 \wedge j < 2 \rangle$ and $\langle a_n : n \in S_{< i_0} \rangle$ satisfy conditions (i) - (iii). Let $k_{i_0} = |L_0 \cup L_1 \cup S_{< i_0}|$ and let us fix an enumeration

$$P(L_0 \cup L_1 \cup A_{< i_0}) = \{K_r : r < 2^{k_{i_0}}\}. \quad (21)$$

First we define $S_{i_0,0}$. Let $m_{i_0,0} = \max(\{0, 1\} \cup S_{<i_0})$. Since $A \in \mathbb{P}(R)$, for each $r < 2^{k_{i_0}}$ the set $A_{K_r}^{L_0 \cup L_1 \cup A_{<i_0}}$ is infinite and, by (14), intersects infinitely many sets L_n . So, for $r < 2^{k_{i_0}}$ let

$$n_{i_0,0}^r = \begin{cases} \min\{n > m_{i_0,0} : A_{K_0}^{L_0 \cup L_1 \cup A_{<i_0}} \cap L_n \neq \emptyset\} & \text{if } r = 0, \\ \min\{n > n_{i_0,0}^{r-1} : A_{K_r}^{L_0 \cup L_1 \cup A_{<i_0}} \cap L_n \neq \emptyset\} & \text{if } r > 0, \end{cases}$$

let us define $S_{i_0,0} = \{n_{i_0,0}^r : r < 2^{k_{i_0}}\}$ and for $r < 2^{k_{i_0}}$ let us choose

$$a_{n_{i_0,0}^r} \in A_{K_r}^{L_0 \cup L_1 \cup A_{<i_0}} \cap L_{n_{i_0,0}^r}. \quad (22)$$

Now we define $S_{i_0,1}$. Let $m_{i_0,1} = \max(\{0, 1\} \cup S_{<i_0} \cup S_{i_0,0})$. For $r < 2^{k_{i_0}}$ let

$$n_{i_0,1}^r = \begin{cases} \min\{n > m_{i_0,1} : A_{K_0}^{L_0 \cup L_1 \cup A_{<i_0}} \cap L_n \neq \emptyset\} & \text{if } r = 0, \\ \min\{n > n_{i_0,1}^{r-1} : A_{K_r}^{L_0 \cup L_1 \cup A_{<i_0}} \cap L_n \neq \emptyset\} & \text{if } r > 0, \end{cases}$$

let us define $S_{i_0,1} = \{n_{i_0,1}^r : r < 2^{k_{i_0}}\}$ and for $r < 2^{k_{i_0}}$ let us choose

$$a_{n_{i_0,1}^r} \in A_{K_r}^{L_0 \cup L_1 \cup A_{<i_0}} \cap L_{n_{i_0,1}^r}. \quad (23)$$

By (21), (22) and (23), the extended sequences $\langle S_{i,j} : 2 \leq i < i_0 + 1 \wedge j < 2 \rangle$ and $\langle a_n : n \in S_{<i_0+1} \rangle$ satisfy conditions (i) and (iii). By the construction we have $S_{<i_0} < S_{i_0,0} < S_{i_0,1}$ and (ii) is true as well. The recursion works indeed. \square

Now we define the sets A_0 and A_1 by:

$$A_0 = L_0 \cup \{a_n : n \in \bigcup_{2 \leq i < \omega} S_{i,0}\} \text{ and } A_1 = L_1 \cup \{a_n : n \in \bigcup_{2 \leq i < \omega} S_{i,1}\}. \quad (24)$$

By (14) and (i) we have $A_0 \cup A_1 \subset A$. By (8) we have $L_0 \cap L_1 = F \cap f^{-1}[G] = \emptyset$, which, together with (i), (ii) and (24), implies $A_0 \cap A_1 = \emptyset$. By (24) we have $F = L_0 \subset A_0$ and $f^{-1}[G] = L_1 \subset A_1$ so $G \subset f[A_1]$ and (15) is true.

We prove (16) showing that the set $A_0 \cup \{p\} \cup f[A_1]$ satisfies (1). Let

$$K_0 \subset H_0 \in [A_0]^{<\omega} \text{ and } K_1 \subset H_1 \in [f[A_1]]^{<\omega}. \quad (25)$$

Since $f^{-1}[K_1] \subset f^{-1}[H_1] \subset A_1$, by (24) there is $i_0 > 2$ such that

$$K_0 \cup f^{-1}[K_1] \subset H_0 \cup f^{-1}[H_1] \subset L_0 \cup L_1 \cup A_{<i_0}. \quad (26)$$

Claim 1. $A_0 \cap R_{\{p\} \cup K_0 \cup K_1}^{\{p\} \cup H_0 \cup H_1} \neq \emptyset$.

Proof of Claim 1. By (26) and (iii) there is $n \in S_{i_0,0}$ such that

$$a_n \in R_{K_0 \cup f^{-1}[K_1]}^{L_0 \cup L_1 \cup A_{<i_0}}. \quad (27)$$

Subclaim 1.1 $a_n \in R_{K_0}^{H_0}$.

Proof of Subclaim 1.1 For $u \in K_0$, by (27) we have $a_n \sim u$. For $u \in H_0 \setminus K_0$, by (26) and (27) and since $H_0 \cap f^{-1}[K_1] = \emptyset$, we have $a_n \not\sim u$. \square

Subclaim 1.2 $a_n \in R_{K_1}^{H_1}$.

Proof of Subclaim 1.2 By the definition of B and since $a_n \in A \setminus (L_0 \cup L_1) \subset B$ there are $n_0 \geq 2$ and $K \subset \bigcup_{k \leq n_0} L_k$ such that

$$a_n = q_K^{\bigcup_{k \leq n_0} L_k} \in R_{\{p\}}^{\{p\}} \cap R_K^{\bigcup_{k \leq n_0} L_k} \cap R_{f[K]}^{f[\bigcup_{k \leq n_0} L_k]}. \quad (28)$$

Thus $a_n \in L_{n_0+1}$ which by (i) implies $n = n_0 + 1$ and, since $n \in S_{i_0,0}$, by (ii) we have $S_{<i_0} < \{n_0 + 1\}$ and by (i),

$$L_0 \cup L_1 \cup A_{<i_0} \subset \bigcup_{k \leq n_0} L_k. \quad (29)$$

By (27), (28) and Lemma 2.2(a) we have

$$(L_0 \cup L_1 \cup A_{<i_0}) \cap (\{p\} \cup K \cup f[K]) = (\{p\} \cup \bigcup_{k \leq n_0} L_k \cup f[\bigcup_{k \leq n_0} L_k]) \cap (K_0 \cup f^{-1}[K_1])$$

which, by (26) and (29), gives

$$(L_0 \cup L_1 \cup A_{<i_0}) \cap K = K_0 \cup f^{-1}[K_1]. \quad (30)$$

For $u \in K_1$ we have $f^{-1}(u) \in f^{-1}[K_1]$ and, by (30), $f^{-1}(u) \in K$ that is $u \in f[K]$ and, by (28) we have $a_n \sim u$.

For $u \in H_1 \setminus K_1$ by (25) we have $f^{-1}(u) \in f^{-1}[H_1] \setminus f^{-1}[K_1] \subset A_1$, which implies $f^{-1}(u) \notin K_0$. Thus

$$f^{-1}(u) \notin K_0 \cup f^{-1}[K_1]. \quad (31)$$

Since $f^{-1}(u) \in f[H_1]$, by (26) $f^{-1}(u) \in L_0 \cup L_1 \cup A_{<i_0}$ so, by (31) and (30), $f^{-1}(u) \notin K$ and, hence

$$u \notin f[K]. \quad (32)$$

By (26) and (29) we have $f^{-1}(u) \in \bigcup_{k \leq n_0} L_k$ which implies $u \in f[\bigcup_{k \leq n_0} L_k]$ so, by (32) and (28), $a_n \not\sim u$. \square

Now, since $n \in S_{i_0,0}$ we have $a_n \in A_0$ and, by (28) and Subclaims 1.1 and 1.2, $a_n \in A_0 \cap R_{\{p\} \cup K_0 \cup K_1}^{\{p\} \cup H_0 \cup H_1}$. Claim 1 is proved. \square

Claim 2. $f[A_1] \cap R_{\emptyset \cup K_0 \cup K_1}^{\{p\} \cup H_0 \cup H_1} \neq \emptyset$.

Proof of Claim 2. By (26) and (iii) there is $n \in S_{i_0,1}$ such that

$$a_n \in R_{K_0 \cup f^{-1}[K_1]}^{L_0 \cup L_1 \cup A_{<i_0}}. \quad (33)$$

Subclaim 2.1 $f(a_n) \in R_{K_1}^{H_1}$.

Proof of Subclaim 2.1 For $u \in K_1$ we have $f^{-1}(u) \in f^{-1}[K_1]$ and, by (33), $a_n \sim f^{-1}(u)$. Thus, since f is an isomorphism, $f(a_n) \sim u$.

For $u \in H_1 \setminus K_1$ we have $f^{-1}(u) \in f^{-1}[H_1] \setminus f^{-1}[K_1]$ and, by (26) and (33) we have $a_n \not\sim f^{-1}(u)$. So, since f is an isomorphism $f(a_n) \not\sim u$. \square

Subclaim 2.2 $f(a_n) \in R_{K_0}^{H_0}$.

Proof of Subclaim 2.2 By the definition of B and since $a_n \in A_1 \setminus (L_0 \cup L_1) \subset B$ there are $n_0 \geq 2$ and $K \subset \bigcup_{k \leq n_0} L_k$ such that

$$a_n = q_K^{\bigcup_{k \leq n_0} L_k} \in R_{\{p\}}^{\{p\}} \cap R_K^{\bigcup_{k \leq n_0} L_k} \cap R_{f[K]}^{f[\bigcup_{k \leq n_0} L_k]}. \quad (34)$$

Thus $a_n \in L_{n_0+1}$ which by (i) implies $n = n_0 + 1$ and, since $n \in S_{i_0,1}$, by (ii) we have $S_{<i_0} < \{n_0 + 1\}$ and, by (i),

$$L_0 \cup L_1 \cup A_{<i_0} \subset \bigcup_{k \leq n_0} L_k. \quad (35)$$

By (33), (34) and Lemma 2.2(a),

$$(L_0 \cup L_1 \cup A_{<i_0}) \cap (\{p\} \cup K \cup f[K]) = (\{p\} \cup \bigcup_{k \leq n_0} L_k \cup f[\bigcup_{k \leq n_0} L_k]) \cap (K_0 \cup f^{-1}[K_1])$$

so, by (26) and (35), $(L_0 \cup L_1 \cup A_{<i_0}) \cap K = K_0 \cup f^{-1}[K_1]$. Thus

$$f[L_0 \cup L_1 \cup A_{<i_0}] \cap f[K] = f[K_0] \cup K_1. \quad (36)$$

Now, for $u \in K_0$ we have $f(u) \in f[K_0]$ and, by (36), $f(u) \in f[K]$ which, by (34) gives $a_n \sim f(u)$ and, by (7), $f(a_n) \sim u$.

For $u \in H_0 \setminus K_0$ we have $f(u) \in f[H_0] \setminus f[K_0]$. Since $u \in A_0$ and $f^{-1}[H_1] \subset A_1$ we have $u \notin f^{-1}[H_1]$ and, hence $f(u) \notin H_1 \supset K_1$. Thus $f(u) \notin f[K_0] \cup K_1$. Since $u \in H_0$ by (26) we have $u \in L_0 \cup L_1 \cup A_{<i_0}$ and, by (36), we have $f(u) \notin f[K]$. By (26) and (35), $u \in \bigcup_{k \leq n_0} L_k$ so $f(u) \in f[\bigcup_{k \leq n_0} L_k] \setminus f[K]$ and, by (34), $a_n \not\sim f(u)$, which, by (7), gives $f(a_n) \not\sim u$. Thus $f(a_n) \in R_{K_0}^{H_0}$. \square

Now, since $n \in S_{i_0,1}$ we have $f(a_n) \in f[A_1]$ and, by (34) and Subclaims 2.1 and 2.2, $f(a_n) \in f[A_1] \cap R_{\emptyset \cup K_0 \cup K_1}^{\{p\} \cup H_0 \cup H_1}$. Claim 2 is proved. \square

By Claims 1 and 2 (16) is true and A_0 is a (p, F, G) -extendible copy below A . \square

Lemma 4.5 *For each maximal antichain \mathcal{A} in $\mathbb{P}(R)$ and each finite set $F_0 \subset R$ there is $S \in \mathbb{P}(R)$ containing F_0 and compatible with $\leq 2^{|F_0|}$ elements of \mathcal{A} .*

Proof. We prove the lemma by induction on $|F_0| = k$. For $k = 0$ this is trivial: take $S \in \mathcal{A}$. Suppose that the statement is true for k . Let $|F_0| = k + 1$, $p \in F_0$ and let

$$F = F_0 \cap R_{\{p\}}^{\{p\}} \quad \text{and} \quad G = F_0 \cap R_{\emptyset}^{\{p\}}. \quad (37)$$

By Lemmas 4.3 and 4.4 there is an isomorphism $f : R_{\{p\}}^{\{p\}} \rightarrow R_{\emptyset}^{\{p\}}$ satisfying (7) and (8) and there is a copy $B \in \mathbb{P}(R)$ satisfying $F \cup f^{-1}[G] \subset B \subset R_{\{p\}}^{\{p\}}$ and such that for each copy $A \in \mathbb{P}(R)$ satisfying $F \cup f^{-1}[G] \subset A \subset B$ there are copies A_0 and A_1 satisfying (15) and (16).

Claim 1. $\mathcal{D} = \{C \in \mathbb{P}(B) : \exists A', A'' \in \mathcal{A} \ C \subset A' \cap f^{-1}[A'' \cap R_{\emptyset}^{\{p\}}]\}$ is a dense set in the poset $\langle \mathbb{P}(B), \subset \rangle$ (for each $E \in \mathbb{P}(B)$ there is $C \in \mathcal{D}$ such that $C \subset E$).

Proof of Claim 1. Let $E \in \mathbb{P}(B)$. Since $E \in \mathbb{P}(R)$, by the maximality of \mathcal{A} there are $A' \in \mathcal{A}$ and $C_1 \in \mathbb{P}(R)$ such that $C_1 \subset E \cap A'$. Since f is an isomorphism we have $f[C_1] \in \mathbb{P}(R)$ and, again, there are $A'' \in \mathcal{A}$ and $C_2 \in \mathbb{P}(R)$ such that $C_2 \subset f[C_1] \cap A''$, which implies that for $C = f^{-1}[C_2]$ we have $C \subset C_1 \subset E \subset B$ and, thus $C \in \mathbb{P}(B)$, and $C \subset f^{-1}[A'' \cap R_{\emptyset}^{\{p\}}]$. Since $C \subset C_1 \subset A'$ we have $C \in \mathcal{D}$ and $C \subset C_1 \subset E$. \square

Let \mathcal{A}^* be a maximal antichain in the poset $\langle \mathcal{D}, \subset \rangle$.

Claim 2. \mathcal{A}^* is a maximal antichain in the poset $\langle \mathbb{P}(B), \subset \rangle$.

Proof of Claim 2. By the density of \mathcal{D} , \mathcal{A}^* is an antichain in $\langle \mathbb{P}(B), \subset \rangle$. If $E \in \mathbb{P}(B)$, by Claim 1 there is $C \in \mathcal{D}$ such that $C \subset E$ and, by the maximality of \mathcal{A}^* in \mathcal{D} , there are $A \in \mathcal{A}^*$ and $C_1 \in \mathcal{D}$ satisfying $C_1 \subset C \cap A \subset E \cap A$. Thus each $E \in \mathbb{P}(B)$ is compatible with some element of \mathcal{A}^* . \square

Since $B \cong R$ (which implies $\mathbb{P}(B) \cong \mathbb{P}(R)$) and since $F \cup f^{-1}[G] \in [B]^k$ and \mathcal{A}^* is a maximal antichain in $\mathbb{P}(B)$, by the induction hypothesis applied to B there is a set A satisfying

$$F \cup f^{-1}[G] \subset A \in \mathbb{P}(B) \quad (38)$$

and compatible with $m \leq 2^k$ elements of \mathcal{A}^* , say C_1, \dots, C_m . Thus

$$\forall C \in \mathcal{A}^* \setminus \{C_1, \dots, C_m\} \quad A \perp C. \quad (39)$$

Since $\mathcal{A}^* \subset \mathcal{D}$, there are sets $A'_1, A''_1, \dots, A'_m, A''_m \in \mathcal{A}$ such that

$$\forall i \leq m \quad C_i \subset A'_i \cap f^{-1}[A''_i \cap R_{\emptyset}^{\{p\}}]. \quad (40)$$

By (38) and Lemma 4.4 there are sets A_0 and A_1 satisfying

$$A_0 \cup A_1 \subset A \quad \wedge \quad A_0 \cap A_1 = \emptyset \quad \wedge \quad F \subset A_0 \quad \wedge \quad G \subset f[A_1], \quad (41)$$

$$S := A_0 \cup \{p\} \cup f[A_1] \in \mathbb{P}(R). \quad (42)$$

By (41) and (42) we have $F_0 = F \cup \{p\} \cup G \subset S$ and it remains to be proved that S is compatible with $\leq 2m(\leq 2^{k+1})$ -many elements of \mathcal{A} . Since $A_0, A_1 \subset A$, by (39) we have

$$\forall C \in \mathcal{A}^* \setminus \{C_1, \dots, C_m\} \quad (A_0 \perp C \wedge A_1 \perp C) \quad (43)$$

and the proof will be finished when we show that

$$\forall D \in \mathcal{A} \setminus \{A'_1, A''_1, \dots, A'_m, A''_m\} \quad S \perp D. \quad (44)$$

On the contrary, suppose that there are $D \in \mathcal{A} \setminus \{A'_1, A''_1, \dots, A'_m, A''_m\}$ and $C \in \mathbb{P}(R)$ such that

$$C \subset S \cap D. \quad (45)$$

By (42), (45) and since R is strongly indivisible (see Fact 2.1(b)), at least one of the sets $C^0 = C \cap A_0$ and $C^1 = C \cap f[A_1]$ is a copy of R .

If $C^0 \in \mathbb{P}(R)$, then, since $C^0 \subset A_0 \subset A \subset B$ and \mathcal{A}^* is a maximal antichain in $\mathbb{P}(B)$, there is $C^* \in \mathcal{A}^*$ such that $C^* \not\perp C^0$, which implies $C^* \not\perp A_0$ thus, by (43), $C^* = C_i$, for some $i \leq m$. By (40) we have $C^* \subset A'_i$ and, since $C^0 \subset C \subset D$ and $C^* \not\perp C^0$, we have $A'_i \not\perp D$, which implies $D = A'_i$. But this contradicts our assumption concerning D .

If $C^1 \in \mathbb{P}(R)$, then, since $C^1 \subset f[A_1]$, we have $f^{-1}[C^1] \subset A_1 \subset B$ and, since f is an isomorphism, $f^{-1}[C^1] \in \mathbb{P}(B)$. Since \mathcal{A}^* is a maximal antichain in $\mathbb{P}(B)$ there is $C^* \in \mathcal{A}^*$ such that $C^* \not\perp f^{-1}[C^1]$ and, since $f^{-1}[C^1] \subset A_1$, we have $C^* \not\perp A_1$. Thus, by (43), $C^* = C_i$, for some $i \leq m$. By (45) we have $C^1 \subset D$ and, since $C^1 \subset R_{\emptyset}^{\{p\}}$, we have $f^{-1}[C^1] \subset f^{-1}[D \cap R_{\emptyset}^{\{p\}}]$ which implies

$$C_i = C^* \not\perp f^{-1}[D \cap R_{\emptyset}^{\{p\}}]. \quad (46)$$

By (40), $C_i \subset f^{-1}[A''_i \cap R_{\emptyset}^{\{p\}}]$ so, by (46), $f^{-1}[D \cap R_{\emptyset}^{\{p\}}] \not\perp f^{-1}[A''_i \cap R_{\emptyset}^{\{p\}}]$ and, hence, there is $E \in \mathbb{P}(R)$ such that $E \subset f^{-1}[D \cap R_{\emptyset}^{\{p\}}] \cap f^{-1}[A''_i \cap R_{\emptyset}^{\{p\}}]$. But then $\mathbb{P}(R) \ni f[E] \subset D \cap A''_i$ and we have $D \not\perp A''_i$, which implies $D = A''_i$. This is a contradiction. Thus (44) is true and the proof is finished. \square

Proof of Theorem 4.2. Let $F_0 \in [R]^{<\omega}$ and let \mathcal{A} be a maximal antichain in $\mathbb{P}(R)$. First we prove that

$$\mathcal{D} = \{C \in \mathbb{P}(R) : \exists A \in \mathcal{A} \exists H \subset F_0 \quad C \subset A \cap R_H^{F_0}\}$$

is a dense set in the poset $\mathbb{P}(R)$. If $B \in \mathbb{P}(R)$, then, by Fact 2.1(a) and (c), $B \setminus F_0 \in \mathbb{P}(R)$ and $B \setminus F_0 = \bigcup_{H \subset F_0} B \cap R_H^{F_0}$ so, since $B \setminus F_0$ is strongly indivisible (Fact 2.1(b)), there is $H_0 \subset F_0$ such that $B \cap R_{H_0}^{F_0} \in \mathbb{P}(R)$. By the maximality of A there are $A_0 \in \mathcal{A}$ and $C \in \mathbb{P}(R)$ such that $C \subset B \cap R_{H_0}^{F_0} \cap A_0$. Thus $C \in \mathcal{D}$ and $C \subset B$.

Let \mathcal{A}^* be a maximal antichain in the poset $\langle \mathcal{D}, \subset \rangle$. Clearly \mathcal{A}^* is a maximal antichain in the poset $\langle \mathbb{P}(R), \subset \rangle$ and, by Lemma 4.5, there is $S \in \mathbb{P}(R)$ containing F_0 and compatible with $m \leq 2^{|F_0|}$ elements of \mathcal{A}^* , say C_1, \dots, C_m . Next we prove that

$$\forall H \subset F_0 \exists i \leq m (C_i \subset R_H^{F_0} \wedge C_i \not\subset S_H^{F_0}). \quad (47)$$

Let $H \subset F_0$. Since $S_H^{F_0} \in \mathbb{P}(R)$ there is $C \in \mathcal{A}^*$ such that $C \not\subset S_H^{F_0}$, which implies $C \not\subset S$ and, hence, $C = C_{i_H}$, for some $i_H \leq m$. Since $C_{i_H} \in \mathcal{D}$ there is $H' \subset F_0$ such that $C_{i_H} \subset R_{H'}^{F_0}$ and, since C_{i_H} is compatible with $S_H^{F_0} \subset R_H^{F_0}$, we have $R_H^{F_0} \not\subset R_{H'}^{F_0}$, which implies $H' = H$ and $C_{i_H} \subset R_H^{F_0}$. Thus, since $m \leq 2^{|F_0|}$, $H \mapsto i_H$ is a bijection from $P(F_0)$ to $\{1, 2, \dots, m\}$ and (47) is true.

Now we prove

$$S_1 = F_0 \cup \bigcup_{H \subset F_0} (C_{i_H} \cap S_H^{F_0}) \in \mathbb{P}(R). \quad (48)$$

Suppose that $S_1 \notin \mathbb{P}(R)$. Then $S \setminus S_1 = \bigcup_{H \subset F_0} (S_H^{F_0} \setminus C_{i_H}) \in \mathbb{P}(R)$ (S is strongly indivisible) and, by the maximality of \mathcal{A}^* , there are $C^* \in \mathcal{A}^*$ and $C \in \mathbb{P}(R)$ such that $C \subset C^* \cap (S \setminus S_1) = \bigcup_{H \subset F_0} C^* \cap (S_H^{F_0} \setminus C_{i_H})$. Thus, since C is strongly indivisible, there is $H_0 \subset F_0$ such that $C_1 = C \cap C^* \cap (S_{H_0}^{F_0} \setminus C_{i_{H_0}}) \in \mathbb{P}(R)$, which implies that S is compatible with $C^* \in \mathcal{A}^* \setminus \{C_{i_H} : H \subset F_0\}$. But, by (47), $\{C_{i_H} : H \subset F_0\} = \{C_i : i \leq m\}$, a contradiction. Thus (48) is true.

Finally we prove (6). For $H \subset F_0$ we have $C_{i_H} \in \mathcal{A}^* \subset \mathcal{D}$ and, hence, there is $A \in \mathcal{A}$ such that $C_{i_H} \subset A$. Thus by (48), $(S_1)_H^{F_0} = C_{i_H} \cap S_H^{F_0} \subset A$. \square

5 Fusion for $\mathbb{P}(R)$

If $\langle R, \sim \rangle$ is the Rado graph and $\mathcal{D} = \langle \mathcal{D}_n : n \in \omega \rangle$ a sequence of subsets of $\mathbb{P}(R)$, then a copy $L \in \mathbb{P}(R)$ will be called a *fusion of \mathcal{D}* if and only if there exists a labeling $\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \rangle$ of L such that

$$\forall n \in \omega \quad \forall K \subset \bigcup_{i < n} L_i \quad \exists D \in \mathcal{D}_n \quad L_K^{\bigcup_{i < n} L_i} \subset D. \quad (49)$$

Theorem 5.1 *If $\mathcal{D} = \langle \mathcal{D}_n : n \in \omega \rangle$ is a sequence subsets of $\mathbb{P}(R)$ which are dense below $A \in \mathbb{P}(R)$, then the set $\mathcal{F} = \{L : L \text{ is a fusion of } \mathcal{D}\}$ is dense below A .*

Proof. Let $B \in \mathbb{P}(R)$ and $B \subset A$. In order to construct an $L \in \mathcal{F} \cap \mathbb{P}(B)$ by recursion we define a sequence $\langle \langle S_n, L_n \rangle : n \in \omega \rangle$ such that for each $n \in \omega$

- (i) $S_n \in \mathbb{P}(B)$,
- (ii) $S_{n+1} \subset S_n$,
- (iii) $\bigcup_{i \leq n} L_i \subset S_n$,
- (iv) $L_n = \{q_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$, where $q_K^{\bigcup_{i < n} L_i} \in (S_n)_K^{\bigcup_{i < n} L_i}$, and
- (v) $\forall K \subset \bigcup_{i < n} L_i \exists D \in \mathcal{D}_n (S_n)_K^{\bigcup_{i < n} L_i} \subset D$.

Since $B \subset A$ the set \mathcal{D}_0 is dense below B . We choose $S_0 \in \mathcal{D}_0$ such that $S_0 \subset B$, take $q_\emptyset^\emptyset \in S_0$, define $L_0 = \{q_\emptyset^\emptyset\}$ and conditions (i) - (v) are satisfied.

Suppose that a sequence $\langle \langle S_i, L_i \rangle : i < n \rangle$ satisfies conditions (i) - (v). Then $S_{n-1} \in \mathbb{P}(B)$ and, hence, the set $\mathcal{D}'_n = \{D \in \mathcal{D}_n : D \subset S_{n-1}\}$ is dense below S_{n-1} . Let \mathcal{A}_n be a maximal antichain in $\langle \mathcal{D}'_n, \subset \rangle$. Clearly \mathcal{A}_n is a maximal antichain in the poset $\langle \mathbb{P}(S_{n-1}), \subset \rangle$ and, by (iii), $\bigcup_{i < n} L_i \subset S_{n-1}$ so, by Theorem 4.2 applied to S_{n-1} , there is a set S_n satisfying

$$\bigcup_{i < n} L_i \subset S_n \in \mathbb{P}(S_{n-1}), \text{ and} \quad (50)$$

$$\forall K \subset \bigcup_{i < n} L_i \exists D \in \mathcal{A}_n (S_n)_K^{\bigcup_{i < n} L_i} \subset D. \quad (51)$$

By (50) conditions (i) and (ii) are satisfied and, since $S_n \cong R$, for $K \subset \bigcup_{i < n} L_i$ we choose $q_K^{\bigcup_{i < n} L_i} \in (S_n)_K^{\bigcup_{i < n} L_i}$ and define $L_n = \{q_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$, so (iii) and (iv) are satisfied too. By (51), for $K \subset \bigcup_{i < n} L_i$ there is $D \in \mathcal{A}_n \subset \mathcal{D}_n$ such that $(S_n)_K^{\bigcup_{i < n} L_i} \subset D$. Thus (v) is true and the recursion works.

We show that $L := \bigcup_{n \in \omega} L_n \in \mathcal{F}$. By (i) we have $S_n \subset A$ and, by (iv), for $K \subset \bigcup_{i < n} L_i$ we have $q_K^{\bigcup_{i < n} L_i} \in (S_n)_K^{\bigcup_{i < n} L_i} \subset A_K^{\bigcup_{i < n} L_i}$ and, by Lemma 4.1(b), $\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} L_i\} \rangle$ is a labeling of L . By (ii) and (iii), for $n \in \omega$ and $K \subset \bigcup_{i < n} L_i$ we have $L \subset S_n$, which together with (v) implies that there is $D \in \mathcal{D}_n$ such that $L_K^{\bigcup_{i < n} L_i} \subset (S_n)_K^{\bigcup_{i < n} L_i} \subset D$ and (49) is true as well. Thus $L \in \mathcal{F}$ and, by (i) and (ii), $L \subset B$; so, \mathcal{F} is dense below A . \square

The following statement is an improvement of Theorem 4.2.

Corollary 5.2 *For each sequence $\mathcal{A} = \{\mathcal{A}_n : n \in \omega\}$ of maximal antichains in the poset $\mathbb{P}(R)$ there is a maximal antichain \mathcal{A}' in $\mathbb{P}(R)$ consisting of fusions of \mathcal{A} .*

Proof. By the assumption, the sets $\mathcal{D}_n = \{D \subset R : \exists A \in \mathcal{A}_n D \subset A\}$, $n \in \omega$, are dense in $\mathbb{P}(R)$ and, by Theorem 5.1, the corresponding set of fusions \mathcal{F} is a dense set as well. If \mathcal{A}' is a maximal chain in \mathcal{F} it is a maximal chain in $\mathbb{P}(R)$. \square

6 New reals and a factorization

Clearly, the $\mathbb{P}(R)$ -name $\rho = \{\langle \check{p}, R_{\{p\}}^{\{p\}} \rangle : p \in R\}$ is a name for a subset of R and, since $|R| = \omega$, ρ can be regarded as a name for a real.

Theorem 6.1 *The name ρ codes a new real, that is, $R \Vdash \rho \in P(\check{R}) \setminus V$.*

Proof. Let G be a $\mathbb{P}(R)$ -generic filter over V . Suppose that $\rho_G = S$ for some $S \in P(R) \cap V$. Then $A \Vdash \rho = \check{S}$, for some $A \in G$, which implies that $A \Vdash \check{p} \in \rho$, for all $p \in S$, and $A \Vdash \check{p} \notin \rho$, for all $p \in R \setminus S$. Since $A \Vdash \check{p} \in \rho$ iff $A \cap R_{\emptyset}^{\{p\}} \in \mathcal{I}_R$ and $A \Vdash \check{p} \notin \rho$ iff $A \cap R_{\{p\}}^{\{p\}} \in \mathcal{I}_R$ we have

$$\forall p \in S \ (A \cap R_{\emptyset}^{\{p\}} \in \mathcal{I}_R) \ \wedge \ \forall p \in R \setminus S \ (A \cap R_{\{p\}}^{\{p\}} \in \mathcal{I}_R) \quad (52)$$

Let $p \in A$. If $p \in S$, then, since $p \in A \in \mathbb{P}(R)$, we have $\mathbb{P}(R) \ni A_{\emptyset}^{\{p\}} = A \cap R_{\emptyset}^{\{p\}}$, which is impossible by (52). If $p \in R \setminus S$, then, $\mathbb{P}(R) \ni A_{\{p\}}^{\{p\}} = A \cap R_{\{p\}}^{\{p\}}$, which is impossible by (52). A contradiction. \square

If G is a $\mathbb{P}(R)$ -generic filter over the ground model V (of ZFC), then, by Theorem 6.1, $\rho_G \notin V$ and (see [6], p. 265) there is a forcing \mathbb{P} and a \mathbb{P} -generic filter over V , H , such that $V[\rho_G] = V_{\mathbb{P}}[H]$. Thus (see [7], p. 48) there is a \mathbb{P} -name for a poset π such that the generic extension $V_{\mathbb{P}(R)}[G]$ is equal to the two-step extension $(V_{\mathbb{P}}[H])_{\pi_H}[H_1] = (V[\rho_G])_{\pi_H}[H_1]$, where H_1 is a π_H -generic filter over $V_{\mathbb{P}}[H]$. In the sequel we show that π is a name for an ω -distributive forcing.

Theorem 6.2 *Let κ be an infinite cardinal and G a $\mathbb{P}(R)$ -generic filter over the ground model V . If $x \in V_{\mathbb{P}(R)}[G]$, where $x : \omega \rightarrow \kappa$, then $x \in V[\rho_G]$.*

Proof. Let τ be a $\mathbb{P}(R)$ -name such that $x = \tau_G$. Then there is $A \in G$ such that $A \Vdash \tau : \check{\omega} \rightarrow \check{\kappa}$ and first we prove that

$$\forall B \in \mathbb{P}(A) \ \exists L \in \mathbb{P}(B) \ L \Vdash \tau \in V[\rho]. \quad (53)$$

Let $B \in \mathbb{P}(A)$. Since $A \Vdash \forall n \in \check{\omega} \ \exists \alpha \in \check{\kappa} \ \tau(\check{n}) = \check{\alpha}$, for each $n \in \omega$ we have: for each $C \in \mathbb{P}(A)$ there are $D \in \mathbb{P}(C)$ and $\alpha \in \kappa$ such that $D \Vdash \tau(\check{n}) = \check{\alpha}$. This means that the sets $\mathcal{D}_n := \{D \in \mathbb{P}(A) : \exists \alpha \in \kappa \ D \Vdash \tau(\check{n}) = \check{\alpha}\}$, $n \in \omega$, are dense below A . By Theorem 5.1, the set \mathcal{F} of fusions is dense below A and, hence, there is $L = \bigcup_{n \in \omega} L_n \in \mathcal{F}$ such that $L \in \mathbb{P}(B)$. By (49), for $n \in \omega$ and $K \subset \bigcup_{i < n} L_i$ there is $D \in \mathcal{D}_n$ such that $L_K^{\bigcup_{i < n} L_i} \subset D$ and, hence, there is (clearly unique) $\alpha \in \kappa$ such that $L_K^{\bigcup_{i < n} L_i} \Vdash \tau(\check{n}) = \check{\alpha}$. Thus we obtain a family

of ordinals $\{\alpha_K^{\bigcup_{i<n} L_i} : n \in \omega \wedge K \subset \bigcup_{i<n} L_i\}$ indexed by elements $q_K^{\bigcup_{i<n} L_i}$ of L such that

$$L_K^{\bigcup_{i<n} L_i} \Vdash \tau(\check{n}) = \alpha_K^{\bigcup_{i<n} L_i}. \quad (54)$$

In order to prove that $L \Vdash \tau \in V[\rho]$ we assume that H is a $\mathbb{P}(R)$ -generic filter over V containing L and we reconstruct τ_H inside $V[\rho_H]$ showing that for each $n \in \omega$

$$\tau_H(n) = \alpha_{(\bigcup_{i<n} L_i) \cap \rho_H}^{\bigcup_{i<n} L_i},$$

which will, by (54), follow from $L_{(\bigcup_{i<n} L_i) \cap \rho_H}^{\bigcup_{i<n} L_i} \in H$. Clearly $R_{\{p\}}^{\{p\}} \in H$, for each $p \in (\bigcup_{i<n} L_i) \cap \rho_H$, and $R_{\emptyset}^{\{p\}} \in H$, for each $p \in (\bigcup_{i<n} L_i) \setminus \rho_H$. Thus

$$R_{(\bigcup_{i<n} L_i) \cap \rho_H}^{\bigcup_{i<n} L_i} = \bigcap_{p \in (\bigcup_{i<n} L_i) \cap \rho_H} R_{\{p\}}^{\{p\}} \cap \bigcap_{p \in (\bigcup_{i<n} L_i) \setminus \rho_H} R_{\emptyset}^{\{p\}} \in H$$

and, since $L \in H$, we have $L_{(\bigcup_{i<n} L_i) \cap \rho_H}^{\bigcup_{i<n} L_i} = L \cap R_{(\bigcup_{i<n} L_i) \cap \rho_H}^{\bigcup_{i<n} L_i} \in H$. So $\tau_H \in V[\rho_H]$ and we proved that $L \Vdash \tau \in V[\rho]$, which completes the proof of (53).

Now, since $A \in G$, by (53) there is $L \in G$ satisfying $L \Vdash \tau \in V[\rho]$ and, hence, $x = \tau_G \in V[\rho_G]$. \square

7 The \aleph_0 -covering and the Sacks property

For a cardinal $\kappa \geq \omega$ and a sequence of positive integers $\langle k_n : n \in \omega \rangle$ a mapping $s : \omega \rightarrow [\kappa]^{<\omega}$ will be called an $\langle k_n \rangle$ -*slalom* in κ iff $|s(n)| \leq k_n$, for each $n \in \omega$. $\text{Sl}_{\langle k_n \rangle}(\kappa)$ will denote the set of all such mappings.

A pre-order \mathbb{P} has the *Sacks property* iff there is a sequence $\langle k_n \rangle \in \mathbb{N}^\omega$ such that in each generic extension $V_{\mathbb{P}}[G]$ for each $x : \omega \rightarrow \omega$ there is $s \in V \cap \text{Sl}_{\langle k_n \rangle}(\omega)$ (or, equivalently, $s \in V \cap \text{Sl}_{\langle 2^n \rangle}(\omega)$) such that $x(n) \in s(n)$, for each $n \in \omega$.

A pre-order \mathbb{P} has the \aleph_0 -*covering property* iff in each generic extension $V_{\mathbb{P}}[G]$ each countable set X of ordinals is contained in a countable set $A \in V$.

We note that the Cohen forcing has the \aleph_0 -covering property (it is a ccc poset) but does not have the Sacks property, while, under CH, the Namba forcing has the Sacks property (since it does not produce new reals) but does not have the \aleph_0 -covering (since it adds a cofinal mapping from ω to ω_2 , see [6]); the Sacks forcing has both of these properties and we show that the same holds for the forcing $\mathbb{P}(R)$.

We recall that a complete Boolean algebra \mathbb{B} is weakly distributive iff for each cardinal κ in each generic extension $V_{\mathbb{B}}[G]$ for each function $x : \omega \rightarrow \kappa$ there is a slalom $s : \omega \rightarrow [\kappa]^{<\omega}$ belonging to V and such that $x(n) \in s(n)$, for all $n \in \omega$.

Theorem 7.1 (a) If κ is an infinite cardinal and G a $\mathbb{P}(R)$ -generic filter over the ground model V , then for each function $x : \omega \rightarrow \kappa$ belonging to $V_{\mathbb{P}(R)}[G]$ there exists a slalom $s \in V \cap \text{Sl}_{\langle m_n \rangle}(\kappa)$ such that $x(n) \in s(n)$, for each $n \in \omega$.

(b) The forcing $\mathbb{P}(R)$ has the \aleph_0 -covering property and, hence, preserves ω_1 .

(c) The forcing $\mathbb{P}(R)$ has the Sacks property.

(d) The algebra $\text{ro sq } \mathbb{P}(R)$ is a weakly distributive complete Boolean algebra.

Proof. (a) We have to prove that for each $\mathbb{P}(R)$ -name τ

$$R \Vdash \tau : \check{\omega} \rightarrow \check{\kappa} \Rightarrow \exists s \in ((\text{Sl}_{\langle m_n \rangle}(\check{\kappa}))^V)^\sim \forall n \in \check{\omega} \tau(n) \in s(n). \quad (55)$$

Thus, working in V we show that for each $A \in \mathbb{P}(R)$ satisfying $A \Vdash \tau : \check{\omega} \rightarrow \check{\kappa}$ there are $L \in \mathbb{P}(A)$ and $s \in \text{Sl}_{\langle m_n \rangle}(\kappa)$ such that

$$\forall n \in \omega \ L \Vdash \tau(\check{n}) \in \check{s}(\check{n}). \quad (56)$$

First, exactly as in the proof of Theorem 6.2 we find $L = \bigcup_{n \in \omega} L_n \in \mathbb{P}(A)$ such that

$$\forall n \in \omega \ \forall K \subset \bigcup_{i < n} L_i \ \exists_1 \alpha_K^n \in \kappa \ L_K^{\bigcup_{i < n} L_i} \Vdash \tau(\check{n}) = \check{\alpha}_K^n. \quad (57)$$

Let the mapping $s : \omega \rightarrow [\kappa]^{<\omega}$ be defined by $s(n) = \{\alpha_K^n : K \subset \bigcup_{i < n} L_i\}$. Since $\{L_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$ is a maximal antichain below L in $\mathbb{P}(R)$, by (57) we have $L \Vdash \tau(\check{n}) \in \check{s}(\check{n})$ and (56) is true. Finally, $s \in \text{Sl}_{\langle m_n \rangle}(\kappa)$ because $|L_i| = m_i$ and $|s(n)| \leq |P(\bigcup_{i < n} L_i)| = 2^{\sum_{i < n} |L_i|} = 2^{\sum_{i < n} m_i} = m_n$.

(b) If $X \in V_{\mathbb{P}(R)}[G] \cap [\kappa]^\omega$ and $x : \omega \rightarrow X$ is a bijection, then by (a) we have $X = x[\omega] \subset \bigcup_{n \in \omega} s(n) \in V$, because $s \in V$.

(c) By (a) each function $x : \omega \rightarrow \omega$ is contained in an $s \in V \cap \text{Sl}_{\langle m_n \rangle}(\omega)$. \square

8 Tree-ordered copies of the Rado graph

Here we show that each labeling of a copy L of the Rado graph $\langle R, \sim \rangle$ induces a reversed tree order on L in a natural way. This construction will be used in the next section. So, let $\mathcal{L} = \langle \{L_n : n \in \omega\}, q \rangle$ be a labeling of L , that is

$$L_n = \{q_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}, \text{ where } q_K^{\bigcup_{i < n} L_i} \in L_K^{\bigcup_{i < n} L_i}. \quad (58)$$

Using the labeling \mathcal{L} we define the binary relation $\leq_{L, \mathcal{L}}$ (we will write shortly \leq_L) on L by:

$$q_{K'}^{\bigcup_{i < m} L_i} \leq_L q_{K''}^{\bigcup_{i < n} L_i} \Leftrightarrow L_{K'}^{\bigcup_{i < m} L_i} \subset L_{K''}^{\bigcup_{i < n} L_i}. \quad (59)$$

Since $L_\emptyset^\emptyset = L$ we have $q \leq_L q_\emptyset^\emptyset$, for all $q \in L$ and, clearly, the relation \leq_L is reflexive, transitive and, by Lemma 2.2(c) antisymmetric. Thus, $\langle L, \leq_L \rangle$ is a partial

order with the largest element q_\emptyset^\emptyset . Lemma 2.2(d) applied to L gives $L_{K'}^{\bigcup_{i < m} L_i} \subset L_{K''}^{\bigcup_{i < n} L_i}$ if and only if $\bigcup_{i < m} L_i \supset \bigcup_{i < n} L_i$, $K' \supset K''$ and $K' \cap \bigcup_{i < n} L_i = K''$ if and only if $m \geq n$ and $K' \cap \bigcup_{i < n} L_i = K''$. Thus, by (59) we have

$$q_{K'}^{\bigcup_{i < m} L_i} \leq_L q_{K''}^{\bigcup_{i < n} L_i} \Leftrightarrow m \geq n \wedge K' \cap \bigcup_{i < n} L_i = K''. \quad (60)$$

In order to state the following theorem we introduce a convenient notation. For $p, q \in L$ let $p \prec_L q$ denote that p is an immediate predecessor of q in $\langle L, \leq_L \rangle$ and let

$$\text{Ip}_{\langle L, \leq_L \rangle}(q) = \{p \in L : p \prec_L q\}.$$

For $q_K^{\bigcup_{i < n} L_i} \in L_n$ and $K_1 \subset L_n$ let $q_K^{\bigcup_{i < n} L_i} \uparrow_{K_1}^{L_n}$ denote the element $q_{K \cup K_1}^{\bigcup_{i < n+1} L_i}$ of L_{n+1} . For simplicity, the intervals $(p, q]_{\langle L, \leq_L \rangle}$ will be denoted by $(p, q]$.

Theorem 8.1 *For each $n \in \omega$ and $K \subset \bigcup_{i < n} L_i$ in the poset $\langle L, \leq_L \rangle$ we have:*

- (a) $(q_K^{\bigcup_{i < n} L_i}, q_\emptyset^\emptyset] = \{q_{K \cap \bigcup_{i < m} L_i}^{\bigcup_{i < m} L_i} : m < n\}$;
- (b) $\langle L, \leq_L \rangle$ is a reversed tree with the top q_\emptyset^\emptyset and the set L_n is its n -th level;
- (c) $(-\infty, q_K^{\bigcup_{i < n} L_i}] = L_K^{\bigcup_{i < n} L_i}$;
- (d) $\text{Ip}_{\langle L, \leq_L \rangle}(q_K^{\bigcup_{i < n} L_i}) = \{q_K^{\bigcup_{i < n} L_i} \uparrow_{K_1}^{L_n} : K_1 \subset L_n\}$;
- (e) $\langle L, \leq_L \rangle$ is a finitely branching reversed tree without minimal nodes. In fact each element of L_n has $2^{|L_n|} = 2^{m_n}$ immediate predecessors.

Proof. (a) If $q_K^{\bigcup_{i < n} L_i} <_L q_{K'}^{\bigcup_{i < m} L_i}$, then $n \geq m$ and $K' = K \cap \bigcup_{i < m} L_i$. Since $q_K^{\bigcup_{i < n} L_i} \neq_L q_{K'}^{\bigcup_{i < m} L_i}$, we have $m < n$. Thus “ \subset ” is true and “ \supset ” is obvious.

(b) If $m_1 < m_2 < n$, then $q_{K \cap \bigcup_{i < m_2} L_i}^{\bigcup_{i < m_2} L_i} <_L q_{K \cap \bigcup_{i < m_1} L_i}^{\bigcup_{i < m_1} L_i}$ and by (a) the interval $(q_K^{\bigcup_{i < n} L_i}, q_\emptyset^\emptyset]$ is a chain of size n . Thus $\langle L, \leq_L \rangle$ is a reversed tree, $\text{ht}(q_K^{\bigcup_{i < n} L_i}) = n$ and, hence, $L_n = \{q_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$ is the n -th level of $\langle L, \leq_L \rangle$.

(c) If $q_{K'}^{\bigcup_{i < m} L_i} \leq_L q_K^{\bigcup_{i < n} L_i}$, then, by (58) and (59), $q_{K'}^{\bigcup_{i < m} L_i} \in L_{K'}^{\bigcup_{i < m} L_i} \subset L_K^{\bigcup_{i < n} L_i}$. Conversely, if $q_{K'}^{\bigcup_{i < m} L_i} \in L_K^{\bigcup_{i < n} L_i}$, then $q_{K'}^{\bigcup_{i < m} L_i} \in L_m \setminus \bigcup_{i < n} L_i$, which implies $m \geq n$. Since $q_{K'}^{\bigcup_{i < m} L_i} \in R_{K'}^{\bigcup_{i < m} L_i} \cap R_K^{\bigcup_{i < n} L_i}$, by Lemma 2.2 we have $K' \cap \bigcup_{i < n} L_i = K \cap \bigcup_{i < m} L_i$ and, since $K \subset \bigcup_{i < n} L_i \subset \bigcup_{i < m} L_i$ we obtain $K' \cap \bigcup_{i < n} L_i = K$. Thus, by (60), $q_{K'}^{\bigcup_{i < m} L_i} \leq_L q_K^{\bigcup_{i < n} L_i}$.

(d) If $q_F^{\bigcup_{i < p} L_i} \prec_L q_K^{\bigcup_{i < n} L_i}$, then by (60) $p \geq n$ and $F \cap \bigcup_{i < n} L_i = K$. Since $p = n$ would imply the equality, we have $p \geq n+1$. For $p > n+1$ we would have $q_F^{\bigcup_{i < p} L_i} <_L q_{F \cap \bigcup_{i < n+1} L_i}^{\bigcup_{i < n+1} L_i} <_L q_{F \cap \bigcup_{i < n} L_i}^{\bigcup_{i < n} L_i} = q_K^{\bigcup_{i < n} L_i}$ which is not true. Thus $p = n+1$ and $q_F^{\bigcup_{i < p} L_i} = q_{F \cap \bigcup_{i < n+1} L_i}^{\bigcup_{i < n+1} L_i} = q_{K \cup (F \cap L_n)}^{\bigcup_{i < n} L_i \cup L_n} = q_K^{\bigcup_{i < n} L_i} \uparrow_{F \cap L_n}^{L_n}$.

Conversely, for $K_1 \subset L_n$ we have $(K \cup K_1) \cap \bigcup_{i < n} L_i = K$ and, by (60), $q_K^{\bigcup_{i < n} L_i} \restriction_{K_1}^{L_n} = q_{K \cup K_1}^{\bigcup_{i < n+1} L_i} \leq_L q_K^{\bigcup_{i < n} L_i}$. If $q_{K \cup K_1}^{\bigcup_{i < n+1} L_i} \leq_L q_F^{\bigcup_{i < p} L_i} \leq_L q_K^{\bigcup_{i < n} L_i}$, then $n+1 \geq p \geq n$. So, if $p = n+1$, then $K \cup K_1 = (K \cup K_1) \cap \bigcup_{i < n+1} L_i = F$, which implies $q_F^{\bigcup_{i < p} L_i} = q_{K \cup K_1}^{\bigcup_{i < n+1} L_i}$. If $p = n$, then $K = F \cap \bigcup_{i < n} L_i = F \cap \bigcup_{i < p} L_i = F$, thus $q_F^{\bigcup_{i < p} L_i} = q_K^{\bigcup_{i < n} L_i}$. So $(q_{K \cup K_1}^{\bigcup_{i < n+1} L_i}, q_K^{\bigcup_{i < n} L_i}) = \emptyset$, that is $q_K^{\bigcup_{i < n} L_i} \restriction_{K_1}^{L_n} \prec_L q_K^{\bigcup_{i < n} L_i}$. Clearly (e) follows from (d). \square

Thus, by Lemma 3.1 each copy of R has infinitely many labelings and the corresponding induced reversed tree orderings. By Theorem 8.1(c) and Corollary 5.2, we have

Corollary 8.2 *For each sequence $\mathcal{A} = \{\mathcal{A}_n : n \in \omega\}$ of maximal antichains in $\mathbb{P}(R)$ there is a maximal antichain \mathcal{A}' in $\mathbb{P}(R)$ such that each $L \in \mathcal{A}'$ has a labeling such that for each $n \in \omega$ the set $\{(-\infty, q] : q \in \text{Lev}_n \langle L, \leq_L \rangle\}$ refines $\mathcal{A}_n \restriction L$.*

9 Strong subtrees of the ordered Rado graph are large

Let $T = \bigcup_{n \in \omega} \text{Lev}_n(T)$ be a tree of height ω . A subset \mathcal{S} of T is called a *strong subtree* of T iff

- (sst1) \mathcal{S} has the unique root,
- (sst2) There is a set $S = \{n_k : k \in \omega\} \in [\omega]^\omega$ such that $\emptyset \neq \text{Lev}_k(\mathcal{S}) \subset \text{Lev}_{n_k}(T)$, for each $k \in \omega$, (\mathcal{S} is called the *level set* of \mathcal{S}),
- (sst3) If $s \in \text{Lev}_k(\mathcal{S})$, then for each T -immediate successor t of s there is a unique $s_t \in \text{Lev}_{k+1}(\mathcal{S})$ such that $s_t \geq t$.

We will use the following consequence of the Halpern-Läuchli Theorem (see [4], [16]).

Fact 9.1 (Halpern-Läuchli) *If T is a countable finitely branching tree with one root and without maximal nodes, then for each finite coloring of T there is a monochromatic strong subtree of T .*

By (b) and (e) of Theorem 8.1 and the obvious dual of Fact 9.1 for reversed trees we have

Theorem 9.2 *If $\langle R, \sim \rangle$ is the Rado graph, $L \subset R$ a copy of R , \mathcal{L} a labeling of L and \leq_L the corresponding order on L , then for each finite coloring of the set L there is a monochromatic strong reversed subtree \mathcal{S} of the reversed tree $\langle L, \leq_L \rangle$.*

Now we show that each strong reversed subtree of $\langle L, \leq_L \rangle$ contains a copy of R .

Theorem 9.3 Let $\langle R, \sim \rangle$ be the Rado graph, $L \in \mathbb{P}(R)$ a copy of R inside R ,

$$\mathcal{L}_L = \left\langle \{L_n : n \in \omega\}, \{q_K^{\bigcup_{i < n} L_i} : n \in \omega \wedge K \subset \bigcup_{i < n} \Lambda_i\} \right\rangle$$

a labeling of L , \leq_L the corresponding reversed tree order on L and \mathcal{S} a strong reversed subtree of $\langle L, \leq_L \rangle$. Then there is a copy $\Lambda \in \mathbb{P}(R)$ satisfying $\Lambda \subset \mathcal{S}$ and there is a labeling of Λ

$$\mathcal{L}_\Lambda = \left\langle \{\Lambda_k : k \in \omega\}, \{p_H^{\bigcup_{j < k} \Lambda_j} : k \in \omega \wedge H \subset \bigcup_{j < k} \Lambda_j\} \right\rangle$$

such that the orders \leq_Λ and \leq_L coincide on Λ and for $k \in \omega$ and $H \subset \bigcup_{j < k} \Lambda_j$,

$$\Lambda_H^{\bigcup_{j < k} \Lambda_j} = (-\infty, p_H^{\bigcup_{j < k} \Lambda_j}]_{\langle \Lambda, \leq_\Lambda \rangle} = \Lambda \cap (-\infty, p_H^{\bigcup_{j < k} \Lambda_j}]_{\langle L, \leq_L \rangle}. \quad (61)$$

Proof. Let $S = \{n_k : k \in \omega\}$ be the level set of \mathcal{S} , where $n_0 < n_1 < n_2 \dots$. For $q \in \text{Lev}_k(\mathcal{S}) = \mathcal{S} \cap L_{n_k}$ and $p \in \text{Ip}_{\langle L, \leq_L \rangle}(q)$ let $\pi_{\mathcal{S}}(p)$ denote the unique element of $\text{Lev}_{k+1}(\mathcal{S}) = \mathcal{S} \cap L_{n_{k+1}}$ satisfying $\pi_{\mathcal{S}}(p) \leq_L p$ (such an element exists uniquely by the dual of (sst3) for reversed trees).

By recursion for $k \in \omega$ we define $\Lambda_k \subset R$ and $p_H^{\bigcup_{j < k} \Lambda_j} \in R$, $H \subset \bigcup_{j < k} \Lambda_j$, such that

$$(\Lambda 1) \Lambda_k \subset \mathcal{S} \cap L_{n_k},$$

$$(\Lambda 2) \Lambda_k = \{p_H^{\bigcup_{j < k} \Lambda_j} : H \subset \bigcup_{j < k} \Lambda_j\},$$

$$(\Lambda 3) p_H^{\bigcup_{j < k} \Lambda_j} = \pi_{\mathcal{S}}(p_{H \cap \bigcup_{j < k-1} \Lambda_j}^{\bigcup_{j < k-1} \Lambda_j} \upharpoonright_{H \cap L_{n_{k-1}}}^{L_{n_{k-1}}}), \text{ if } k \geq 1 \text{ and } H \subset \bigcup_{j < k} \Lambda_j.$$

First we prove that the recursion works. By the duals of (sst1) and (sst2) \mathcal{S} has the unique top and $\text{Lev}_0(\mathcal{S}) = \mathcal{S} \cap L_{n_0} = \{p\}$, for some p . So for $p_\emptyset^\emptyset := p$ and $\Lambda_0 := \{p_\emptyset^\emptyset\}$ the sequence $\langle \Lambda_0 \rangle$ satisfies $(\Lambda 1)$ - $(\Lambda 3)$.

Suppose that a sequence $\langle \Lambda_0, \dots, \Lambda_k \rangle$ satisfies $(\Lambda 1)$ - $(\Lambda 3)$. Let $H \subset \bigcup_{j < k+1} \Lambda_j$. Then $H = (H \cap \bigcup_{j < k} \Lambda_j) \cup (H \cap \Lambda_k)$ and, by the assumption,

$$p_{H \cap \bigcup_{j < k} \Lambda_j}^{\bigcup_{j < k} \Lambda_j} \in \Lambda_k \subset \mathcal{S} \cap L_{n_k} \text{ and } H \cap \Lambda_k \subset L_{n_k}, \quad (62)$$

$$H \cap \bigcup_{j < k} \Lambda_j \subset \bigcup_{i < n_k} L_i \quad (63)$$

and, by (L3) for \mathcal{L}_L we have $p_{H \cap \bigcup_{j < k} \Lambda_j}^{\bigcup_{j < k} \Lambda_j} = q_K^{\bigcup_{i < n_k} L_i}$, for some $K \subset \bigcup_{i < n_k} L_i$. So, by (62) and Theorem 8.1(d) we have

$$p_{H \cap \bigcup_{j < k} \Lambda_j}^{\bigcup_{j < k} \Lambda_j} \upharpoonright_{H \cap L_{n_k}}^{L_{n_k}} = q_K^{\bigcup_{i < n_k} L_i} \upharpoonright_{H \cap L_{n_k}}^{L_{n_k}} \in \text{Ip}_{\langle L, \leq_L \rangle}(p_{H \cap \bigcup_{j < k} \Lambda_j}^{\bigcup_{j < k} \Lambda_j}) \subset L_{n_{k+1}}$$

and, by (sst3) the element $p_H^{\bigcup_{j<k+1} \Lambda_j} := \pi_S(p_{H \cap \bigcup_{j<k} \Lambda_j} \upharpoonright_{H \cap L_{n_k}}^{L_{n_k}})$ is well defined and belongs to $\mathcal{S} \cap L_{n_{k+1}}$. Thus defining $\Lambda_{k+1} := \{p_H^{\bigcup_{j<k+1} \Lambda_j} : H \subset \bigcup_{j<k+1} \Lambda_j\}$ we have $\Lambda_{k+1} \subset \mathcal{S} \cap L_{n_{k+1}}$ and the sequence $\langle \Lambda_0, \dots, \Lambda_k, \Lambda_{k+1} \rangle$ satisfies conditions (Λ1) - (Λ3). The recursion works indeed.

In order to prove that Λ is a copy of R and \mathcal{L}_Λ its labeling, using induction we show that

$$\forall k \in \omega \quad \forall H \subset \bigcup_{j<k} \Lambda_j \quad p_H^{\bigcup_{j<k} \Lambda_j} \in R_H^{\bigcup_{j<k} \Lambda_j}. \quad (64)$$

For $k = 0$ we have $p_\emptyset^\emptyset \in R_\emptyset^\emptyset = R$.

Suppose that $k \in \omega$ and $p_H^{\bigcup_{j<k} \Lambda_j} \in R_H^{\bigcup_{j<k} \Lambda_j}$, for all $H \subset \bigcup_{j<k} \Lambda_j$, and let $H' \subset \bigcup_{j<k+1} \Lambda_j$. Then $H' = (H' \cap \bigcup_{j<k} \Lambda_j) \cup (H' \cap \Lambda_k)$ and we show that

$$p_{H'}^{\bigcup_{j<k+1} \Lambda_j} \in R_{H'}^{\bigcup_{j<k+1} \Lambda_j} = R_{H' \cap \bigcup_{j<k} \Lambda_j}^{\bigcup_{j<k} \Lambda_j} \cap R_{H' \cap \Lambda_k}^{\Lambda_k}. \quad (65)$$

By (Λ1), (Λ1), and the induction hypothesis we have

$$L_{n_k} \ni p_{H' \cap \bigcup_{j<k} \Lambda_j}^{\bigcup_{j<k} \Lambda_j} \in R_{H' \cap \bigcup_{j<k} \Lambda_j}^{\bigcup_{j<k} \Lambda_j} \quad (66)$$

thus, by (L3) and (L4) for \mathcal{L}_L ,

$$p_{H' \cap \bigcup_{j<k} \Lambda_j}^{\bigcup_{j<k} \Lambda_j} = q_K^{\bigcup_{i<n_k} L_i} \in R_K^{\bigcup_{i<n_k} L_i}, \text{ where } K \subset \bigcup_{i<n_k} L_i. \quad (67)$$

By (66), (67) and Lemma 2.2(a) we have $H' \cap \bigcup_{j<k} \Lambda_j \cap \bigcup_{i<n_k} L_i = K \cap \bigcup_{j<k} \Lambda_j$ and, by (Λ1), $\bigcup_{j<k} \Lambda_j \subset \bigcup_{j<k} L_{n_j} \subset \bigcup_{i<n_k} L_i$. Thus we have

$$H' \cap \bigcup_{j<k} \Lambda_j = K \cap \bigcup_{j<k} \Lambda_j. \quad (68)$$

By (Λ3) and (67) $p_{H'}^{\bigcup_{j<k+1} \Lambda_j} = \pi_S(p_{H' \cap \bigcup_{j<k} \Lambda_j} \upharpoonright_{H' \cap L_{n_k}}^{L_{n_k}}) = \pi_S(q_{K \cup (H' \cap L_{n_k})}^{\bigcup_{i \leq n_k} L_i}) = q_F^{\bigcup_{i < n_{k+1}} L_i}$, where $F \subset \bigcup_{i < n_{k+1}} L_i$ and, hence $q_F^{\bigcup_{i < n_{k+1}} L_i} \leq_L q_{K \cup (H' \cap L_{n_k})}^{\bigcup_{i \leq n_k} L_i}$, which, by (60), implies

$$F \cap (\bigcup_{i < n_k} L_i \cup L_{n_k}) = K \cup (H' \cap L_{n_k}). \quad (69)$$

Since $\bigcup_{j<k} \Lambda_j \subset \bigcup_{i < n_k} L_i$ by (68) and (69) we obtain $F \cap \bigcup_{j<k} \Lambda_j = K \cap \bigcup_{j<k} \Lambda_j = H' \cap \bigcup_{j<k} \Lambda_j$. Thus

$$p_{H'}^{\bigcup_{j<k+1} \Lambda_j} = q_F^{\bigcup_{i < n_{k+1}} L_i} \in R_F^{\bigcup_{i < n_{k+1}} L_i} \subset R_{F \cap \bigcup_{j<k} \Lambda_j}^{\bigcup_{j<k} \Lambda_j} = R_{H' \cap \bigcup_{j<k} \Lambda_j}^{\bigcup_{j<k} \Lambda_j}. \quad (70)$$

By (A1), $\Lambda_k \subset L_{n_k}$, by (67) we have $K \cap \Lambda_k = \emptyset$ so, by (69), $F \cap \Lambda_k = H' \cap \Lambda_k$ and, hence, $p_{H'}^{\bigcup_{j < k+1} \Lambda_j} = q_F^{\bigcup_{i < n_{k+1}} L_i} \in R_F^{\bigcup_{i < n_{k+1}} L_i} \subset R_{F \cap \Lambda_k}^{\Lambda_k} = R_{H' \cap \Lambda_k}^{\Lambda_k}$, which, together with (70) gives (65). So $\Lambda \in \mathbb{P}(R)$ and \mathcal{L}_Λ is a labeling of Λ .

By Theorem 8.1, the labelings \mathcal{L}_L and \mathcal{L}_Λ determine the reversed tree orderings \leq_L and \leq_Λ on L and Λ respectively, in the following way:

$$q_{K'}^{\bigcup_{i < m} L_i} \leq_L q_{K''}^{\bigcup_{i < n} L_i} \Leftrightarrow m \geq n \wedge K' \cap \bigcup_{i < n} L_i = K'', \quad (71)$$

$$p_{H'}^{\bigcup_{j < k} \Lambda_j} \leq_\Lambda p_{H''}^{\bigcup_{j < l} \Lambda_j} \Leftrightarrow k \geq l \wedge H' \cap \bigcup_{j < l} \Lambda_j = H''. \quad (72)$$

and the sets L_i , $i \in \omega$, and Λ_j , $j \in \omega$, are the corresponding levels. In order to show that $\leq_\Lambda = \leq_L \cap \Lambda^2$, using induction we prove that for each $k \in \mathbb{N}$ we have

$$\forall u, v \in \bigcup_{j < k} \Lambda_j \quad (u <_\Lambda v \Leftrightarrow u <_L v). \quad (73)$$

For $k = 1$ this follows from $|\Lambda_0| = 1$. Let $k \in \mathbb{N}$ and suppose that (73) is true. We show that

$$\forall u, v \in \bigcup_{j < k} \Lambda_j \cup \Lambda_k \quad (u <_\Lambda v \Leftrightarrow u <_L v). \quad (74)$$

Let $u, v \in \bigcup_{j < k} \Lambda_j \cup \Lambda_k$.

(\Rightarrow) Let $u <_\Lambda v$. If $u, v \in \bigcup_{j < k} \Lambda_j$, then, by (73), $u <_L v$. Otherwise, since Λ_i 's are the levels of the reversed tree $\langle \Lambda, \leq_\Lambda \rangle$, we have $u \in \Lambda_k$ and $v \in \Lambda_l$, for some $l < k$. Also, there is $w \in \Lambda_{k-1}$ such that $u <_\Lambda w \leq_\Lambda v$ and, since (73) gives $w \leq_L v$ it remains to be shown that $u <_L w$. By (A3) $u = p_H^{\bigcup_{j < k} \Lambda_j} = \pi_S(p_{H \cap \bigcup_{j < k-1} \Lambda_j}^{\bigcup_{j < k-1} \Lambda_j} \uparrow^{L_{n_{k-1}}} p_{H \cap L_{n_{k-1}}}^{\bigcup_{j < k-1} \Lambda_j}) \leq_L p_{H \cap \bigcup_{j < k-1} \Lambda_j}^{\bigcup_{j < k-1} \Lambda_j} \uparrow^{L_{n_{k-1}}} p_{H \cap L_{n_{k-1}}}^{\bigcup_{j < k-1} \Lambda_j} <_L p_{H \cap \bigcup_{j < k-1} \Lambda_j}^{\bigcup_{j < k-1} \Lambda_j}$. But, by (72) we have $u = p_H^{\bigcup_{j < k} \Lambda_j} <_\Lambda p_{H \cap \bigcup_{j < k-1} \Lambda_j}^{\bigcup_{j < k-1} \Lambda_j} \in \Lambda_{k-1}$, which implies that $p_{H \cap \bigcup_{j < k-1} \Lambda_j}^{\bigcup_{j < k-1} \Lambda_j} = w$ and, hence, $u <_L w$.

(\Leftarrow) Let $u <_L v$. If $u, v \in \bigcup_{j < k} \Lambda_j$, then, by (73), $u <_\Lambda v$. Otherwise, since $\Lambda_k \subset L_{n_k}$ and L_n 's are the levels of the reversed tree $\langle L, \leq_L \rangle$, we have $u \in \Lambda_k$ and $v \in \Lambda_l$, for some $l < k$. Then, since \mathcal{L}_L and \mathcal{L}_Λ are labelings of L and Λ ,

$$u = p_{H'}^{\bigcup_{j < k} \Lambda_j} = q_F^{\bigcup_{i < n_k} L_i}, \text{ where } H' \subset \bigcup_{j < k} \Lambda_j \text{ and } F \subset \bigcup_{i < n_k} L_i, \quad (75)$$

$$v = p_{H''}^{\bigcup_{j < l} \Lambda_j} = q_G^{\bigcup_{i < n_l} L_i}, \text{ where } H'' \subset \bigcup_{j < l} \Lambda_j \text{ and } G \subset \bigcup_{i < n_l} L_i, \quad (76)$$

so, by (72), for a proof that $u \leq_\Lambda v$ it remains to be shown that $H' \cap \bigcup_{j < l} \Lambda_j = H''$.

By (75), (76), Lemma 2.2(a) and (71) we have

$$H' \cap \bigcup_{i < n_k} L_i = F \cap \bigcup_{j < k} \Lambda_j \quad \text{and} \quad H'' \cap \bigcup_{i < n_l} L_i = G \cap \bigcup_{j < l} \Lambda_j, \quad (77)$$

$$F \cap \bigcup_{i < n_l} L_i = G. \quad (78)$$

Since $\bigcup_{j < l} \Lambda_j \subset \bigcup_{i < n_l} L_i \subset \bigcup_{i < n_k} L_i$ from (77) and (78) we obtain

$$H' \cap \bigcup_{j < l} \Lambda_j = F \cap \bigcup_{j < l} \Lambda_j \quad \text{and} \quad H'' \cap \bigcup_{j < l} \Lambda_j = G \cap \bigcup_{j < l} \Lambda_j, \quad (79)$$

$$F \cap \bigcup_{j < l} \Lambda_j = G \cap \bigcup_{j < l} \Lambda_j. \quad (80)$$

Now, since $H'' \cap \bigcup_{j < l} \Lambda_j = H''$, the equality $H' \cap \bigcup_{j < l} \Lambda_j = H''$ follows from (79) and (80).

The first equality in (61) follows Theorem 8.1(c) applied to Λ , while the second follows from the equality $\leq_\Lambda = \leq_L \cap \Lambda^2$. \square

10 No splitting reals are added

In this section we show that the poset $\mathbb{P}(R)$ shares one more property with the Sacks forcing. We recall that if \mathbb{P} is a forcing notion and $V_{\mathbb{P}}[G]$ a generic extension of the ground model V by \mathbb{P} , then a real $x \subset \omega$ belonging to $V_{\mathbb{P}}[G]$ is called a *splitting real* iff $|A \cap x| = |A \setminus x| = \omega$ for each infinite set $A \subset \omega$ belonging to V . It is well known that the Sacks forcing does not produce splitting reals and that the same holds for the Miller rational perfect forcing (which does not have the Sacks property). Here we show that the poset $\mathbb{P}(R)$ (and, consequently, the first iterand \mathbb{P} in the two-step iteration $\mathbb{P} * \pi$, see Section 6) has this property as well.

Theorem 10.1 *The forcing $\langle \mathbb{P}(R), \subset \rangle$ does not produce splitting reals.*

Proof. We prove that for each $\mathbb{P}(R)$ -name τ

$$R \Vdash \tau \subset \check{\omega} \Rightarrow \exists S \in (([\omega]^\omega)^V)^\sim (S \subset \tau \vee S \subset \check{\omega} \setminus \tau). \quad (81)$$

Thus, working in V and assuming that $\mathbb{P}(R) \ni A \Vdash \tau \subset \check{\omega}$ it is sufficient to find $\Lambda \in \mathbb{P}(A)$ and $S \in [\omega]^\omega$ such that

$$\Lambda \Vdash \check{S} \subset \tau \vee \Lambda \Vdash \check{S} \subset \check{\omega} \setminus \tau. \quad (82)$$

Since the sets $\mathcal{D}_n = \{D \in \mathbb{P}(R) : D \Vdash \check{n} \in \tau \vee D \Vdash \check{n} \notin \tau\}$, $n \in \omega$, are dense in $\mathbb{P}(R)$, by Theorem 5.1 the set of fusions of the sequence $\langle D_n : n \in \omega \rangle$ is dense as well and, hence, there is a fusion $L = \bigcup_{n \in \omega} L_n \subset A$. So we have $L_n = \{q_K^{\bigcup_{i < n} L_i} : K \subset \bigcup_{i < n} L_i\}$, where $q_K^{\bigcup_{i < n} L_i} \in L_K^{\bigcup_{i < n} L_i}$, and, by (49), for each $n \in \omega$ and each $K \subset \bigcup_{i < n} L_i$ there is $D \in \mathcal{D}_n$ such that $L_K^{\bigcup_{i < n} L_i} \subset D$. Thus

$$\forall n \in \omega \quad \forall K \subset \bigcup_{i < n} L_i \quad (L_K^{\bigcup_{i < n} L_i} \Vdash \check{n} \in \tau \vee L_K^{\bigcup_{i < n} L_i} \Vdash \check{n} \notin \tau). \quad (83)$$

By Theorem 8.1 $\langle L, \leq_L \rangle$ is a reversed tree and for each $K \subset \bigcup_{i < n} L_i$ we have $L_K^{\bigcup_{i < n} L_i} = (-\infty, q_K^{\bigcup_{i < n} L_i}]_{\langle L, \leq_L \rangle}$. So, by (83), $L = L' \cup L''$ is a coloring of L , where

$$L' = \{q_K^{\bigcup_{i < n} L_i} \in L : (-\infty, q_K^{\bigcup_{i < n} L_i}]_{\langle L, \leq_L \rangle} \Vdash \check{n} \in \tau\},$$

$$L'' = \{q_K^{\bigcup_{i < n} L_i} \in L : (-\infty, q_K^{\bigcup_{i < n} L_i}]_{\langle L, \leq_L \rangle} \Vdash \check{n} \notin \tau\}.$$

Now by Theorem 9.2 there is a monochromatic strong reversed subtree \mathcal{S} of the reversed tree $\langle L, \leq_L \rangle$. Let $S = \{n_k : k \in \omega\}$ be the level set of \mathcal{S} .

First suppose that $\mathcal{S} \subset L'$. By Theorem 9.3 there is a copy $\Lambda = \bigcup_{k \in \omega} \Lambda_k \subset \mathcal{S}$ such that $\Lambda_k = \{p_H^{\bigcup_{j < k} \Lambda_j} : H \subset \bigcup_{j < k} \Lambda_j\} \subset L_{n_k}$ and $p_H^{\bigcup_{j < k} \Lambda_j} \in \Lambda_H^{\bigcup_{j < k} \Lambda_j}$.

We prove that $\Lambda \Vdash \check{S} \subset \tau$, that is $\Lambda \Vdash \check{n}_k \in \tau$, for all $k \in \omega$. Since $\{\Lambda_H^{\bigcup_{j < k} \Lambda_j} : H \subset \bigcup_{j < k} \Lambda_j\}$ is an antichain in $\langle \mathbb{P}(R), \subset \rangle$ maximal below Λ , for a proof of $\Lambda \Vdash \check{n}_k \in \tau$ it is sufficient to show that for each $H \subset \bigcup_{j < k} \Lambda_j$ we have

$$\Lambda_H^{\bigcup_{j < k} \Lambda_j} \Vdash \check{n}_k \in \tau. \quad (84)$$

By Theorem 9.3 we have $\Lambda_H^{\bigcup_{j < k} \Lambda_j} = \Lambda \cap (-\infty, p_H^{\bigcup_{j < k} \Lambda_j}]_{\langle L, \leq_L \rangle}$ so

$$\Lambda_H^{\bigcup_{j < k} \Lambda_j} \subset (-\infty, p_H^{\bigcup_{j < k} \Lambda_j}]_{\langle L, \leq_L \rangle}. \quad (85)$$

Since $p_H^{\bigcup_{j < k} \Lambda_j} \in L_{n_k}$ we have $p_H^{\bigcup_{j < k} \Lambda_j} = q_K^{\bigcup_{i < n_k} L_i}$, for some $K \subset \bigcup_{i < n_k} L_i$. Now $\Lambda \subset \mathcal{S} \subset L'$ implies $q_K^{\bigcup_{i < n_k} L_i} \in L'$; thus $(-\infty, q_K^{\bigcup_{i < n_k} L_i}]_{\langle L, \leq_L \rangle} \Vdash \check{n}_k \in \tau$ and, by (85), $\Lambda_H^{\bigcup_{j < k} \Lambda_j} \Vdash \check{n}_k \in \tau$. So (84) is proved.

If $\mathcal{S} \subset L''$, then in a similar way we prove that $\Lambda \Vdash \check{S} \subset \check{\omega} \setminus \tau$. \square

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